# Heteroskedasticity

## EC 320: Introduction to Econometrics

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Winter 2022

# Prologue

# Housekeeping

**Final Exam**

Review lecture this Wednesday.

- Come prepared with questions.

**Exam:** Wednesday, March 16 at 14:45pm (here).

Office hours Tuesday, March 14th at 12:00pm.

**Problem Set 5**

Due today by 11:59pm.

# Heteroskedasticity

# Heteroskedasticity

Let's revisit assumption #4:

> 4. **Homoskedasticity:** The disurbances have **constant variance** $\sigma^2$
>
> ,
>
> - $\boldsymbol{E}\left[u_i^2|X\right] = \text{Var}(u_i|X) = \sigma^2 \implies \text{Var}(u_i) = \sigma^2$ Specifically, we will focus on the assumption of **constant variance** (also known as *homoskedasticity*).

**Violation of this assumption:**

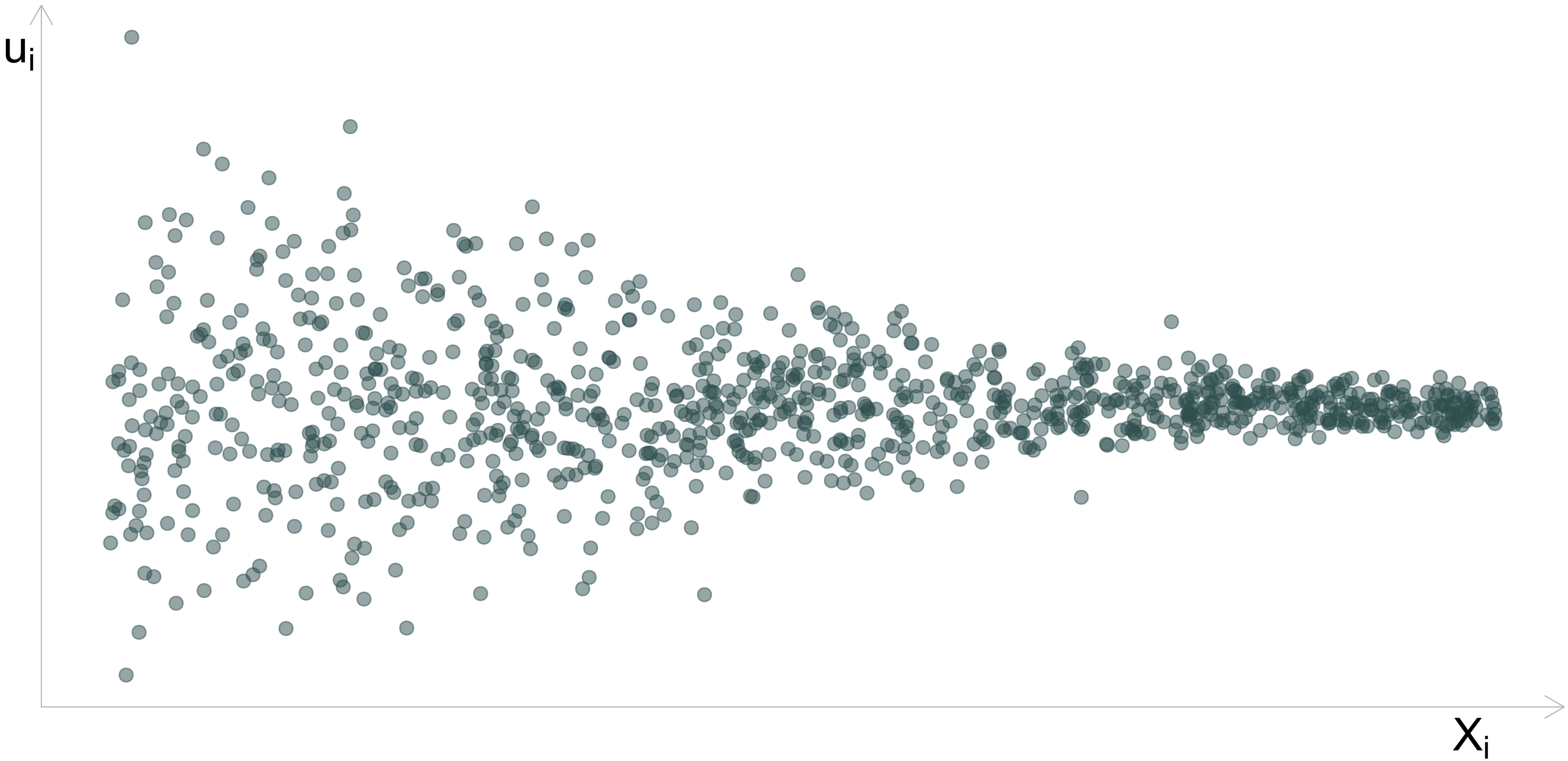
**Heteroskedasticity:** $\text{Var}(u_i) = \sigma_i^2$ and $\sigma_i^2 \neq \sigma_j^2$ for some $i \neq j$.

In other words: Our disturbances have different variances.

# Heteroskedasticity

An easy way to spot this: plot your residuals against your covariates!

For example, variance of $u_i$ decreases with $X_i$, non-constant.



# Heteroskedasticity

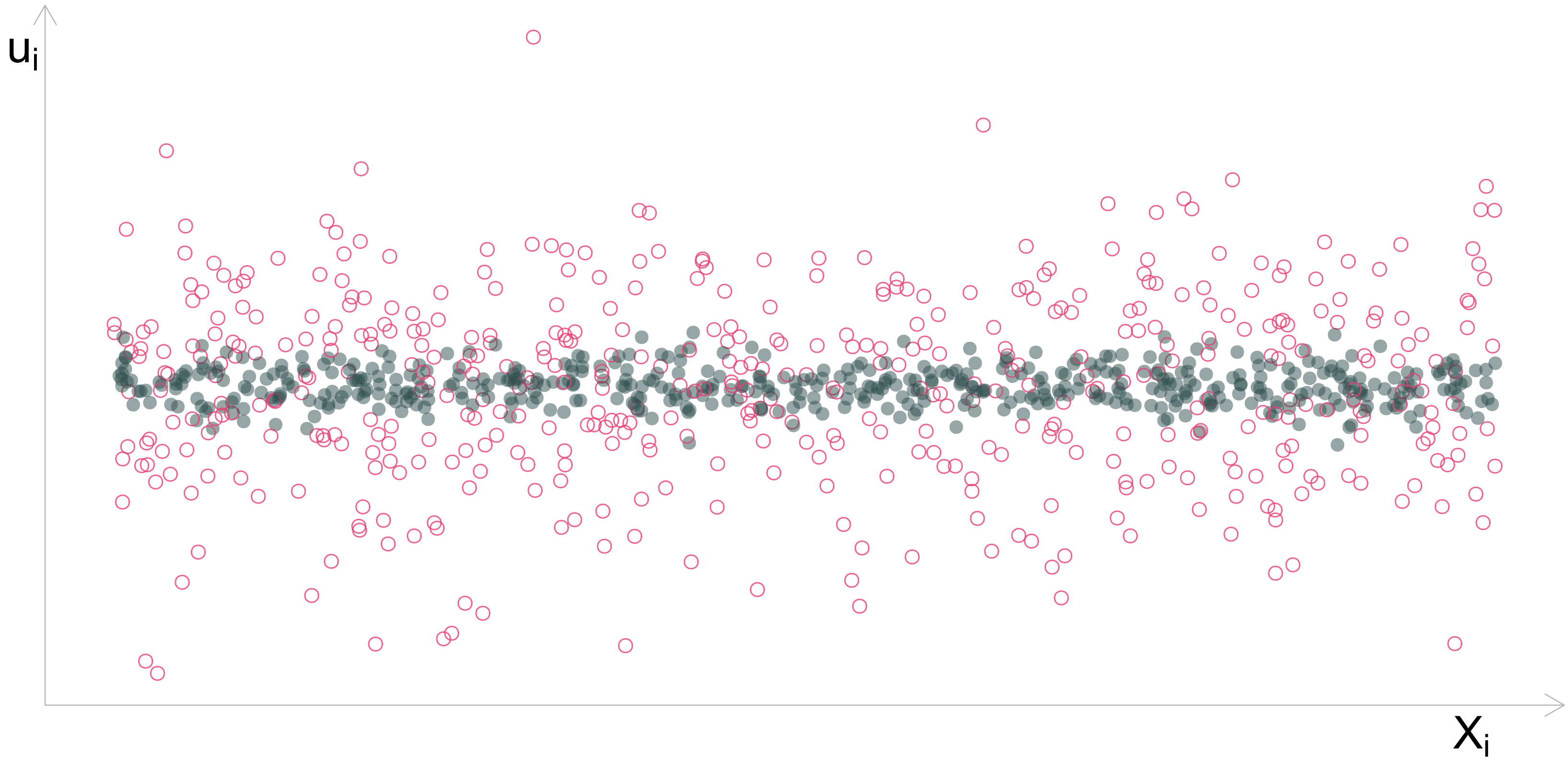
Another example of heteroskedasticity: residuals increase in variation as X deviates further from the mean.

Variance of $u_i$ increasing at the extremes of $X_i$.

# Heteroskedasticity

Another example of heteroskedasticity:

Differing variances of $u_i$ by group



# Heteroskedasticity

**Heteroskedasticity** is present when the variance of $u_i$ changes with any combination of our explanatory variables $X_1$, through $X_j$.

(Very common in practice)

**Why we care:** Heteroskedasticity shows us how small violations of our assumptions can affect OLS's performance.

- If we do not account for heteroskedasticity, our standard error are mispecified

## Consequences

So what are the consquences of heteroskedasticity? Bias? Inefficiency?

First, let's check if it has consquences for the the unbiasedness of OLS.

**Recall$_1$:** OLS being unbiased means $\boldsymbol{E}\left[\hat{\beta}_j \middle| X\right] = \beta_j$ for all $j$ covariates.

**Recall$_2$:** We previously showed $\hat{\beta}_1 = \dfrac{\sum_i \left(y_i - \overline{y}\right)\left(x_{1i} - \overline{x}_1\right)}{\sum_i \left(x_{1i} - \overline{x}_1\right)^2}$

It will actually help us to rewrite this estimator as

$$\hat{\beta}_1 = \beta_1 + \frac{\sum_i \left(x_{1i} - \overline{x}_1\right) u_i}{\sum_i \left(x_{1i} - \overline{x}_1\right)^2}$$

# Heteroskedasticity

**Proof:** Assuming $y_i = \beta_0 + \beta_1 x_i + u_i$

$$
\begin{aligned}
\hat{\beta}_1 &= \frac{\sum_i \left(y_i - \overline{y}\right)\left(x_{1i} - \overline{x}_1\right)}{\sum_i \left(x_{1i} - \overline{x}_1\right)^2} \\
&= \frac{\sum_i \left(\left[\beta_0 + \beta_1 x_{1i} + u_i\right] - \left[\beta_0 + \beta_1 \overline{x}_1 + \overline{u}\right]\right)\left(x_{1i} - \overline{x}_1\right)}{\sum_i \left(x_{1i} - \overline{x}_1\right)^2} \\
&= \frac{\sum_i \left(\beta_1 \left[x_{1i} - \overline{x}_1\right] + \left[u_i - \overline{u}\right]\right)\left(x_{1i} - \overline{x}_1\right)}{\sum_i \left(x_{1i} - \overline{x}_1\right)^2} \\
&= \frac{\sum_i \left(\beta_1 \left[x_{1i} - \overline{x}_1\right]^2 + \left[x_{1i} - \overline{x}_1\right]\left[u_i - \overline{u}\right]\right)}{\sum_i \left(x_{1i} - \overline{x}_1\right)^2} \\
&= \beta_1 + \frac{\sum_i \left(x_{1i} - \overline{x}_1\right)\left(u_i - \overline{u}\right)}{\sum_i \left(x_{1i} - \overline{x}_1\right)^2}
\end{aligned}
$$

# Heteroskedasticity

$$
\begin{aligned}
\hat{\beta}_1 = \cdots &= \beta_1 + \frac{\sum_i \left(x_{1i} - \overline{x}_1\right)\left(u_i - \overline{u}\right)}{\sum_i \left(x_{1i} - \overline{x}_1\right)^2} \\
&= \beta_1 + \frac{\sum_i \left(x_{1i} - \overline{x}_1\right) u_i - \overline{u} \sum_i \left(x_{1i} - \overline{x}_1\right)}{\sum_i \left(x_{1i} - \overline{x}_1\right)^2} \\
&= \beta_1 + \frac{\sum_i \left(x_{1i} - \overline{x}_1\right) u_i - \overline{u} \left(\sum_i x_{1i} - \sum_i \overline{x}_1\right)}{\sum_i \left(x_{1i} - \overline{x}_1\right)^2} \\
&= \beta_1 + \frac{\sum_i \left(x_{1i} - \overline{x}_1\right) u_i - \overline{u} \left(\sum_i x_{1i} - n\overline{x}_1\right)}{\sum_i \left(x_{1i} - \overline{x}_1\right)^2} \\
&= \beta_1 + \frac{\sum_i \left(x_{1i} - \overline{x}_1\right) u_i - \overline{u} \left(\sum_i x_{1i} - \sum_i x_{1i}\right)}{\sum_i \left(x_{1i} - \overline{x}_1\right)^2} \\
&= \beta_1 + \frac{\sum_i \left(x_{1i} - \overline{x}_1\right) u_i}{\sum_i \left(x_{1i} - \overline{x}_1\right)^2} \qquad 😅
\end{aligned}
$$

# Heteroskedasticity

## Consequences: Bias

We now want to see if heteroskedasticity biases the OLS estimator for $\beta_1$.

$$
\begin{aligned}
\boldsymbol{E}\left[\hat{\beta}_1 \middle| X\right] &= \boldsymbol{E}\left[\beta_1 + \frac{\sum_i \left(x_i - \overline{x}\right) u_i}{\sum_i \left(x_i - \overline{x}\right)^2} \middle| X\right] \\
&= \beta_1 + \boldsymbol{E}\left[\frac{\sum_i \left(x_i - \overline{x}\right) u_i}{\sum_i \left(x_i - \overline{x}\right)^2} \middle| X\right] \\
&= \beta_1 + \frac{\sum_i \left(x_i - \overline{x}\right)}{\sum_i \left(x_i - \overline{x}\right)^2} \underbrace{\boldsymbol{E}\left[u_i | X\right]}_{=0} \\
&= \beta_1
\end{aligned}
$$

**OLS is still unbiased** for the $\beta_k$.

## Consequences: Efficiency

OLS's efficiency and inference do not survive heteroskedasticity.

- In the presence of heteroskedasticity, OLS is **no longer the most efficient** (best) linear unbiased estimator.
- It would be more informative (efficient) to **weight observations** inversely to their $u_i$'s variance.
    - Downweight high-variance $u_i$'s (too noisy to learn much).
    - Upweight observations with low-variance $u_i$'s (more 'trustworthy').
    - Now you have the idea of weighted least squares (WLS)

# Heteroskedasticity

## Consequences: Inference

OLS **standard errors are biased** in the presence of heteroskedasticity.

- Wrong confidence intervals
- Problems for hypothesis testing (both $t$ and $F$ tests)
- It's hard to learn much without sound inference.

# Heteroskedasticity

## Solutions

1. **Tests** to determine whether heteroskedasticity is present.

2. **Remedies** for (1) efficiency and (2) inference

# Testing for heteroskedasticity

# Testing for heteroskedasticity

While we *might* have solutions for heteroskedasticity, the efficiency of our estimators depends upon whether or not heteroskedasticity is present.

1. The **Goldfeld-Quandt test**

2. The **Breusch-Pagan test**

3. The **White test**

Each of these tests centers on the fact that we can **use the OLS residual $\hat{u}_i$ to estimate the population disturbance $u_i$**.

## The Goldfeld-Quandt test

Focuses on a specific type of heteroskedasticity: whether the variance of $u_i$ differs **between two groups**.[†]

Remember how we used our residuals to estimate the $\sigma^2$?

$$s^2 = \frac{\text{RSS}}{n-1} = \frac{\sum_i \hat{u}_i^2}{n-1}$$

We will use this same idea to determine whether there is evidence that our two groups differ in the variances of their disturbances, effectively comparing $s_1^2$ and $s_2^2$ from our two groups.

[†]: The G-Q test was one of the early tests of heteroskedasticity (1965).

## The Goldfeld-Quandt test

Operationally,

1. Order your the observations by $x$

2. Split the data into two groups of size n*

    - $G_1$: The first third
    - $G_2$: The last third

3. Run separate regressions of $y$ on $x$ for $G_1$ and $G_2$

4. Record $RSS_1$ and $RSS_2$

5. Calculate the G-Q test statistic

## The Goldfeld-Quandt test

The G-Q test statistic

$$F_{(n^\star-k,\, n^\star-k)} = \frac{\text{RSS}_2/(n^\star-k)}{\text{RSS}_1/(n^\star-k)} = \frac{\text{RSS}_2}{\text{RSS}_1}$$

follows an $F$ distribution (under the null hypothesis) with $n^\star - k$ and $n^\star - k$ degrees of freedom.[†]

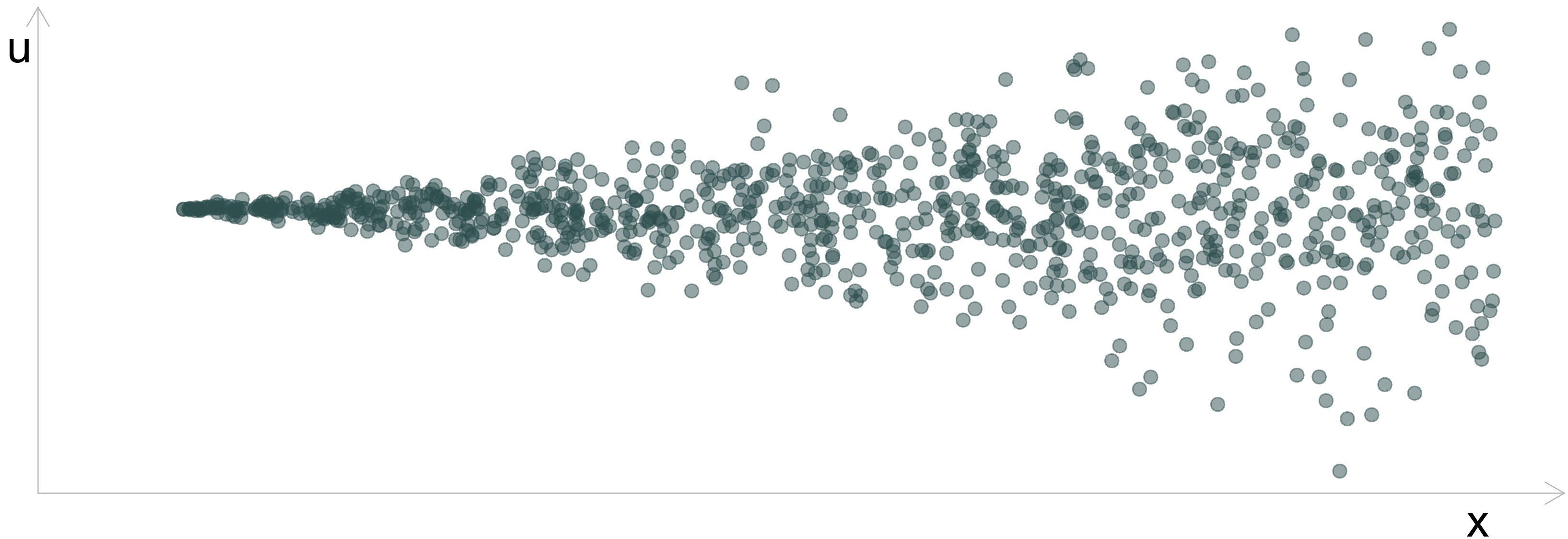
**Notes**

- The G-Q test requires the disturbances follow normal distributions.
- The G-Q assumes a very specific type/form of heteroskedasticity.
- Performs very well if we know the form of potentially heteroskedasticity.

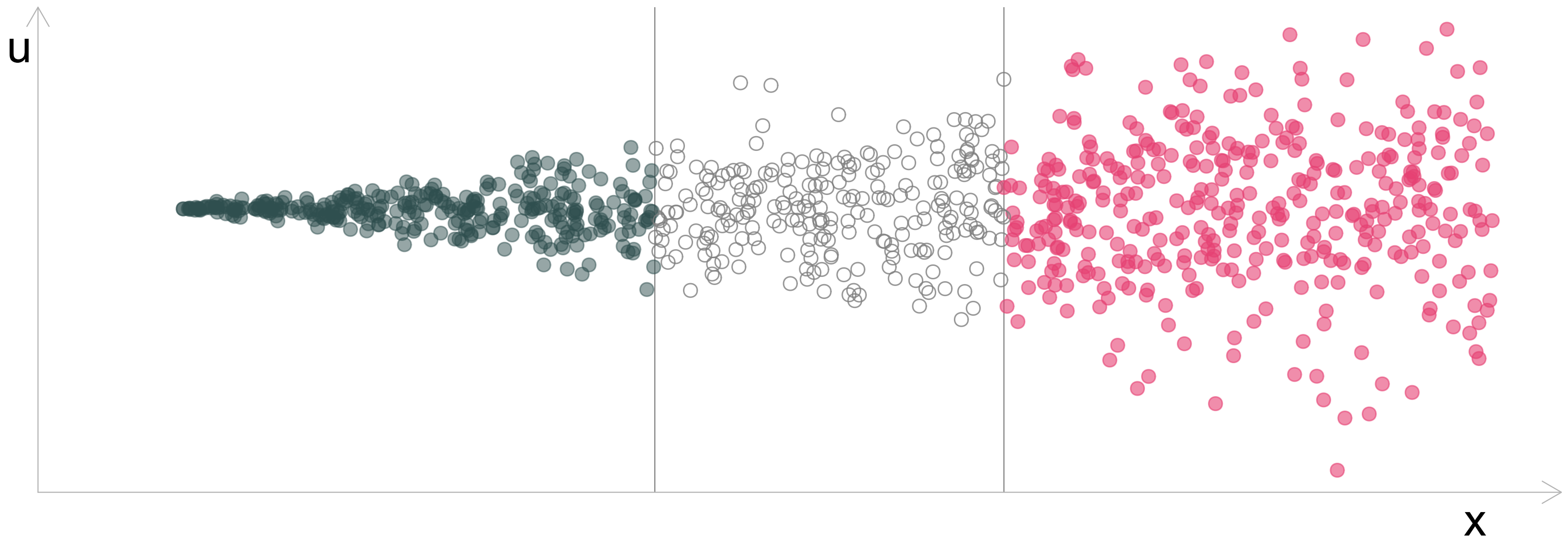
[†]: Goldfeld and Quandt suggested $n^\star$ of $(3/8)n$. $k$ gives number of estimated parameters (*i.e.*, $\hat{\beta}_j$'s).

# Testing for heteroskedasticity

## The Goldfeld-Quandt test



## The Goldfeld-Quandt test

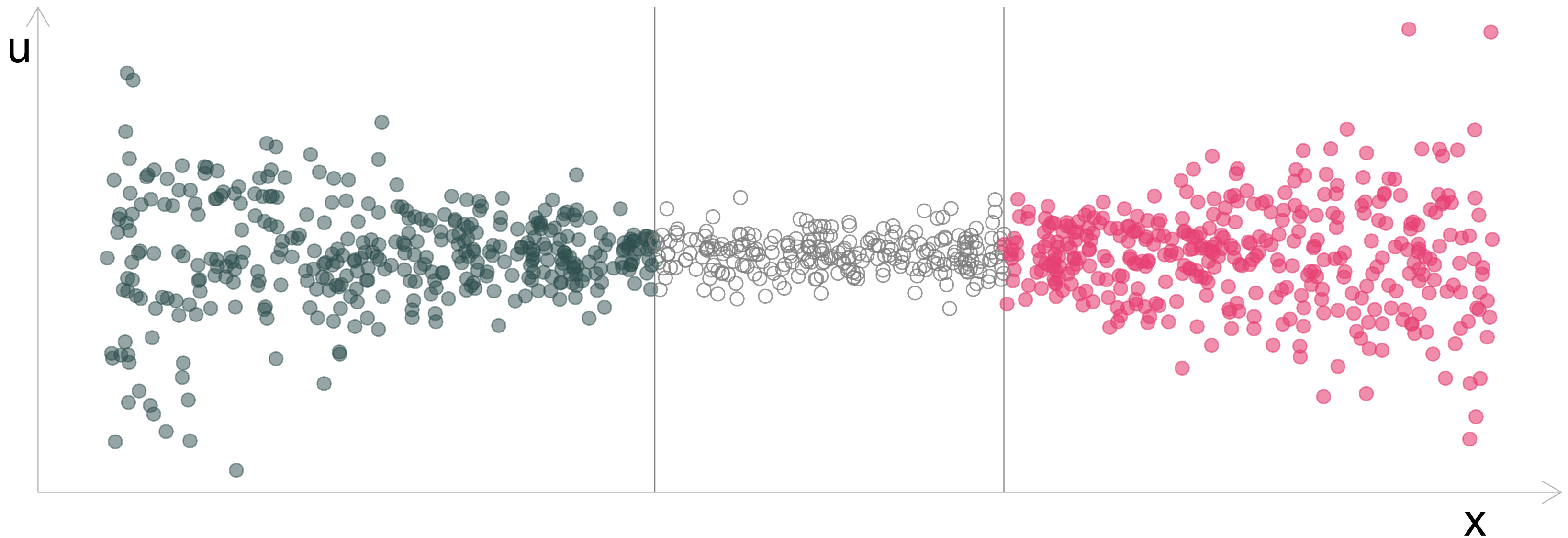



$$F_{375,\,375} = \frac{\text{RSS}_2 = 18,203.4}{\text{RSS}_1 = 1,039.5} \approx 17.5 \implies p\text{-value} < 0.001$$

$\therefore$ We reject $\text{H}_0$: $\sigma_1^2 = \sigma_2^2$ and conclude there is statistically significant evidence of heteroskedasticity.

# Testing for heteroskedasticity

## The Goldfeld-Quandt test




$$F_{375,\,375} = \frac{\text{RSS}_2 = 14,516.8}{\text{RSS}_1 = 14,937.1} \approx 1 \implies p\text{-value} \approx 0.609$$

$\therefore$ We fail to reject H$_0$: $\sigma_1^2 = \sigma_2^2$ while heteroskedasticity is present.

# Testing for heteroskedasticity

## The Breusch-Pagan test

Breusch and Pagan (1981) attempted to solve this issue of being too specific with the functional form of the heteroskedasticity.

- Allows the data to show if/how the variance of $u_i$ correlates with $X$.
- If $\sigma_i^2$ correlates with $X$, then we have heteroskedasticity.
- Regresses $\hat{u}_i^2$ on $X = [1, x_1, x_2, \ldots, x_k]$ and tests for joint significance.

# Testing for heteroskedasticity

## The Breusch-Pagan test

How to implement:

1. Regress y on an intercept, $x_1$, $x_2$, …, $x_k$.

2. Record residuals $\hat{u}_i$.

3. Regress $\hat{u}_i^2$ on an intercept, $x_1$, $x_2$, …, $x_k$.

$$\hat{u}_i^2 = \alpha_0 + \alpha_1 x_{1i} + \alpha_2 x_{2i} + \cdots + \alpha_k x_{ki} + v_i$$

4. Record $R^2$.

5. Test hypothesis $H_0$: $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$

## The Breusch-Pagan test

The B-P test statistic[†] is

$$\mathrm{LM} = n \times R_u^2$$

where $R_u^2$ is the $R^2$ from the regression

$$\hat{u}_i^2 = \alpha_0 + \alpha_1 x_{1i} + \alpha_2 x_{2i} + \cdots + \alpha_k x_{ki} + v_i$$

Under the null, $\mathrm{LM}$ is asymptotically distributed as $\chi_k^2$.

This test statistic tests $\mathrm{H}_0$: $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$.

Rejecting the null hypothesis implies evidence of heteroskedasticity.

[†]: This specific form of the test statistic actually comes form Koenker (1981).

## The $\chi^2$ distribution

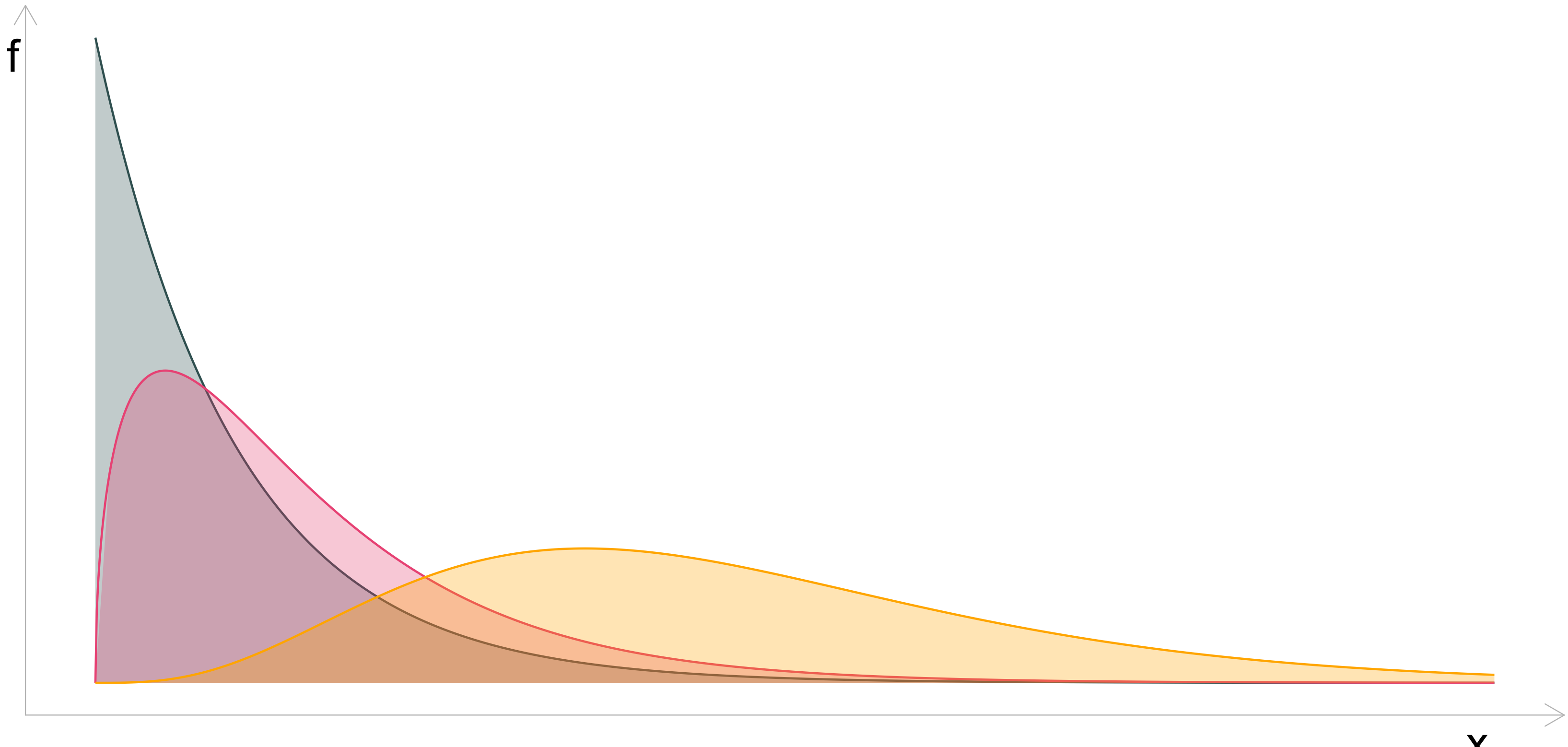
We just mentioned that under the null, the B-P test statistic is distributed as a $\chi^2$ random variable with $k$ degrees of freedom.

The $\chi^2$ distribution is just another example of a common (named) distribution (like the Normal distribution, the $t$ distribution, and the $F$).

# Testing for heteroskedasticity

## The $\chi^2$ distribution

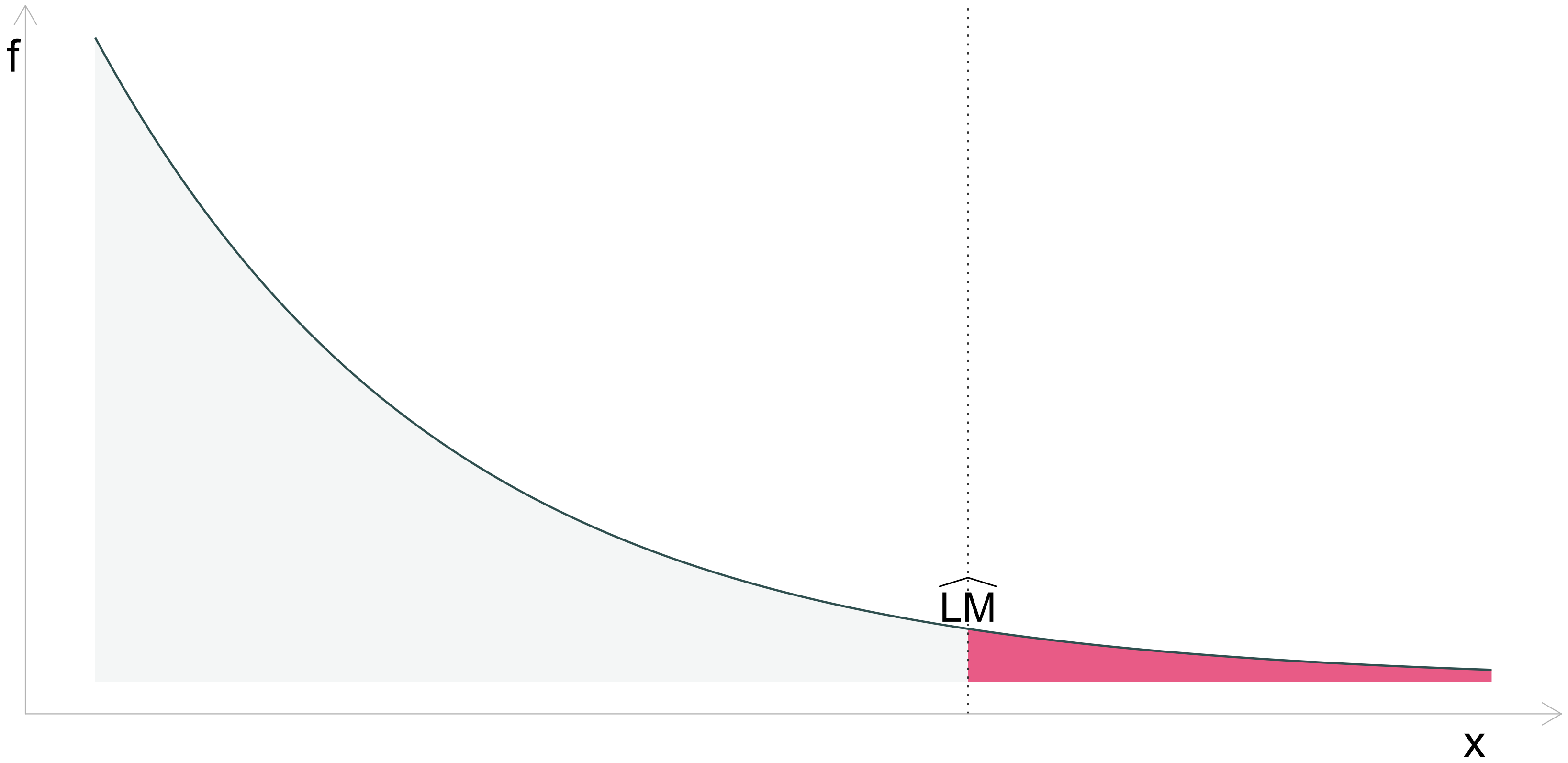
Three examples of $\chi^2_k$: $k = 1$, $k = 2$, and $k = 9$



# Testing for heteroskedasticity

## The $\chi^2$ distribution

Probability of observing a more extreme test statistic $\widehat{\mathrm{LM}}$ under $H_0$



# Testing for heteroskedasticity

## The Breusch-Pagan test

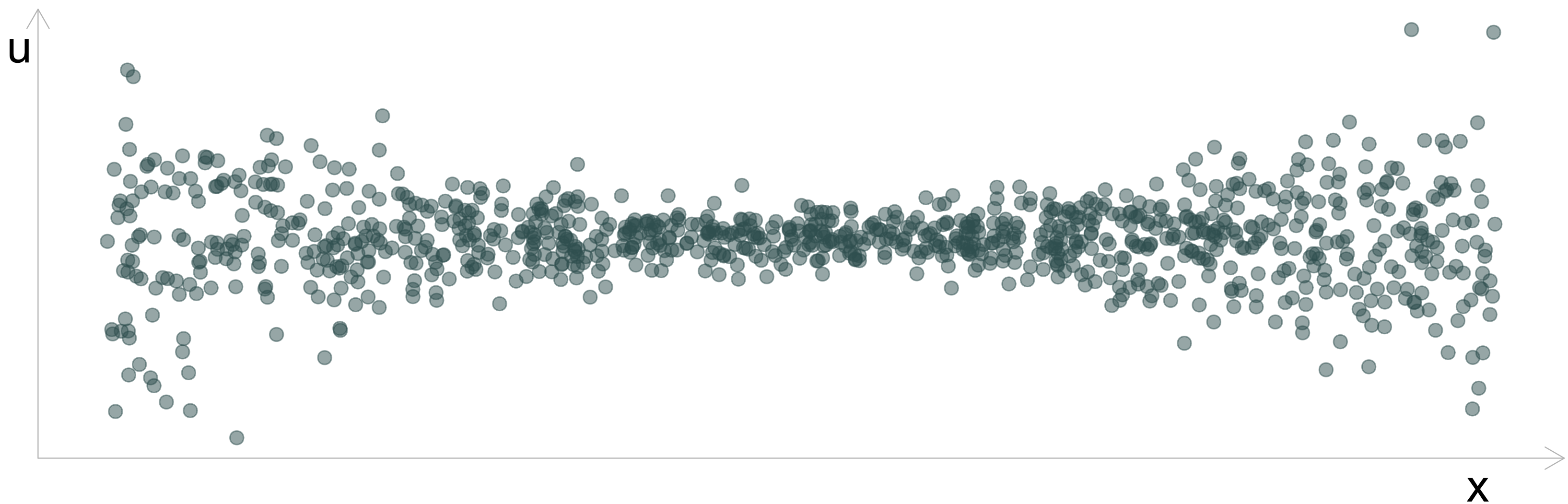
**Problem:** We're still assuming a fairly restrictive **functional form** between our explanatory variables $X$ and the variances of our disturbances $\sigma_i^2$.

**Result:** B-P *may* still miss fairly simple forms of heteroskedasticity.

# Testing for heteroskedasticity

## The Breusch-Pagan test

Breusch-Pagan tests are still **sensitive to functional form**.




$$\hat{u}_i^2 = \hat{\alpha}_0 + \hat{\alpha}_1 x_{1i} \qquad \widehat{\text{LM}} = 1.26 \qquad p\text{-value} \approx 0.261$$

$$\hat{u}_i^2 = \hat{\alpha}_0 + \hat{\alpha}_1 x_{1i} + \hat{\alpha}_2 x_{1i}^2 \qquad \widehat{\text{LM}} = 185.8 \qquad p\text{-value} < 0.001$$

# Testing for heteroskedasticity

## The White test

So far we've been testing for specific relationships between our explanatory variables and the variances of the disturbances, *e.g.*,

- $\text{H}_0$: $\sigma_1^2 = \sigma_2^2$ for two groups based upon $x_j$ (**G-Q**)
- $\text{H}_0$: $\alpha_1 = \cdots = \alpha_k = 0$ from $\hat{u}_i^2 = \alpha_0 + \alpha_1 x_{1i} + \cdots + \alpha_k x_{ki} + v_i$ (**B-P**)

However, we actually want to know if

$$\sigma_1^2 = \sigma_2^2 = \cdots = \sigma_n^2$$

**Q:** Can't we just test this hypothesis? **A:** Sort of.

# Testing for heteroskedasticity

## The White test

Toward this goal, Hal White took advantage of the fact that we can **replace the homoskedasticity requirement with a weaker assumption**:

- **Old:** $\mathrm{Var}(u_i|X) = \sigma^2$
- **New:** $u^2$ is *uncorrelated* with the explanatory variables (*i.e.*, $x_j$ for all $j$), their squares (*i.e.*, $x_j^2$), and the first-degree interactions (*i.e.*, $x_j x_h$).

This new assumption is easier to explicitly test (*hint:* regression).

## The White test

An outline of White's test for heteroskedasticity:

1. Regress y on $x_1$, $x_2$, …, $x_k$. Save residuals e.

2. Regress squared residuals on all explanatory variables, their squares, and interactions.

$$\hat{u}^2 = \alpha_0 + \sum_{h=1}^{k} \alpha_h x_h + \sum_{j=1}^{k} \alpha_{k+j} x_j^2 + \sum_{\ell=1}^{k-1} \sum_{m=\ell+1}^{k} \alpha_{\ell,m} x_\ell x_m + v_i$$

3. Record $R_u^2$.

4. Calculate test statistic to test $H_0$: $\alpha_p = 0$ for all $p \neq 0$.

# Testing for heteroskedasticity

## The White test

Just as with the Breusch-Pagan test, White's test statistic is

$$\text{LM} = n \times R_u^2 \qquad \text{Under H}_0,\ \text{LM} \overset{\text{d}}{\sim} \chi_k^2$$

but now the $R_u^2$ comes from the regression of $\hat{u}_i^2$ on the explanatory variables, their squares, and their interactions.

$$\hat{u}^2 = \alpha_0 + \underbrace{\sum_{h=1}^{k} \alpha_h x_h}_{\text{Expl. variables}} + \underbrace{\sum_{j=1}^{k} \alpha_{k+j} x_j^2}_{\text{Squared terms}} + \underbrace{\sum_{\ell=1}^{k-1} \sum_{m=\ell+1}^{k} \alpha_{\ell,m} x_\ell x_m}_{\text{Interactions}} + v_i$$

**Note:** The $k$ (for our $\chi_k^2$) equals the number of estimated parameters in the regression above (the $\alpha_j$), excluding the intercept ($\alpha_0$).

# Testing for heteroskedasticity

## The White test

**Practical note:** If a variable is equal to its square (*e.g.*, binary variables), then you don't (can't) include it. The same rule applies for interactions.

# Testing for heteroskedasticity

## The White test

*Example:* Consider the model[†] $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + u$

**Step 1:** Estimate the model; obtain residuals ($\hat{u}_i$).

**Step 2:** Regress $\hat{u}^2$ on explanatory variables, squares, and interactions.

$$
\begin{aligned}
\hat{u}^2 = & \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_1^2 + \alpha_5 x_2^2 + \alpha_6 x_3^2 \\
& + \alpha_7 x_1 x_2 + \alpha_8 x_1 x_3 + \alpha_9 x_2 x_3 + v
\end{aligned}
$$

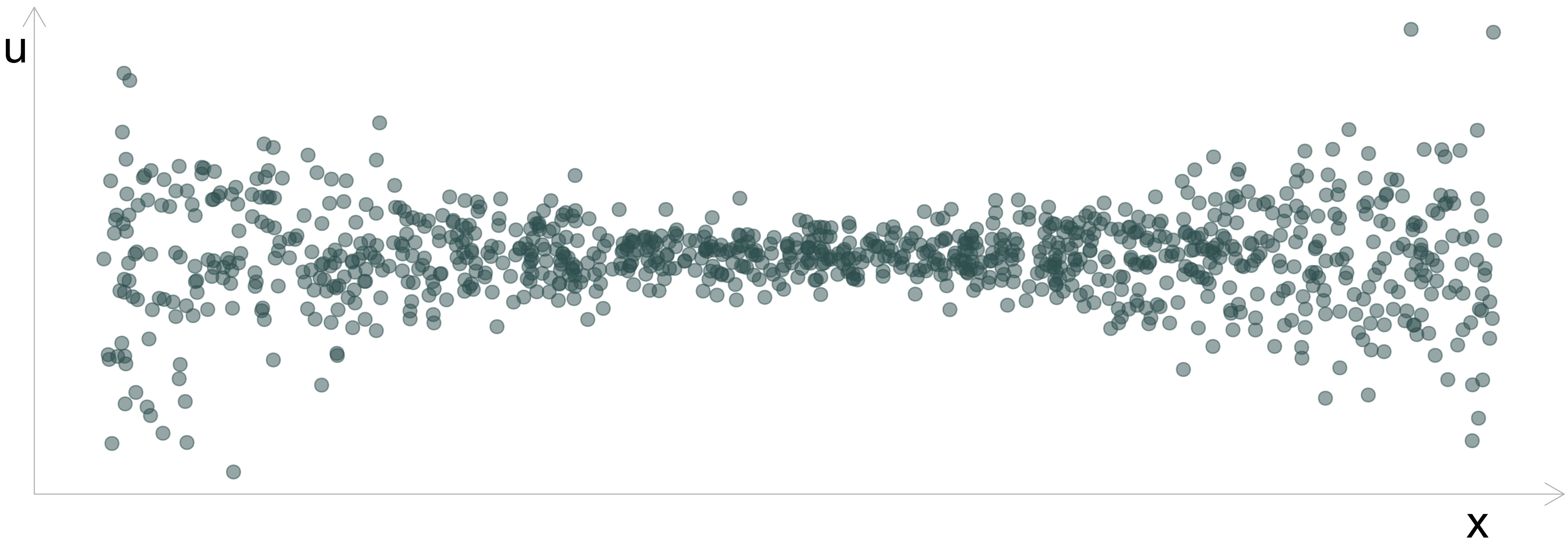
Record the $R^2$ from this equation (call it $R_u^2$).

**Step 3:** Test $H_0$: $\alpha_1 = \alpha_2 = \cdots = \alpha_9 = 0$ using $\text{LM} = n R_u^2 \overset{\text{d}}{\sim} \chi_9^2$.

[†]: To simplify notation here, I'm dropping the $i$ subscripts.

# Testing for heteroskedasticity

## The White test




We've already done the White test for this simple linear regression.

$$\hat{u}_i^2 = \hat{\alpha}_0 + \hat{\alpha}_1 x_{1i} + \hat{\alpha}_2 x_{1i}^2 \qquad \widehat{\text{LM}} = 185.8 \qquad p\text{-value} < 0.001$$

# Testing for Heteroskedasticity

*Examples*

# Testing for heteroskedasticity

## Examples

**Goal:** Estimate the relationship between standardized test scores (outcome variable) and (1) student-teacher ratio and (2) income, *i.e.*,

$$(\textbf{Test score})_i = \beta_0 + \beta_1 \textbf{Ratio}_i + \beta_2 \textbf{Income}_i + u_i \tag{1}$$

**Potential issue:** Heteroskedasticity... and we do not observe $u_i$.

**Solution:**

1. Estimate the relationship in $(1)$ using OLS
2. Use the residuals $(\hat{u}_i)$ to test for heteroskedasticity
   - Goldfeld-Quandt
   - Breusch-Pagan
   - White

## Examples

We will use testing data from the dataset `Caschool` in the `Ecdat` R package.

```r
# Load packages
library(pacman)
p_load(tidyverse, Ecdat)
# Select and rename desired variables; assign to new dataset
test_df ← select(Caschool, test_score = testscr, ratio = str, income = avginc)
# Format as tibble
test_df ← as_tibble(test_df)
# View first 2 rows of the dataset
head(test_df, 2)
```

```
#> # A tibble: 2 x 3
#>   test_score ratio income
#>        <dbl> <dbl>  <dbl>
#> 1       691.  17.9  22.7 
#> 2       661.  21.5   9.82
```

# Testing for heteroskedasticity

## Examples

Let's begin by estimating our model

$$(\text{Test score})_i = \beta_0 + \beta_1 \text{Ratio}_i + \beta_2 \text{Income}_i + u_i$$

```
# Estimate the model
est_model <- lm(test_score ~ ratio + income, data = test_df)
# Summary of the estimate
tidy(est_model)
```

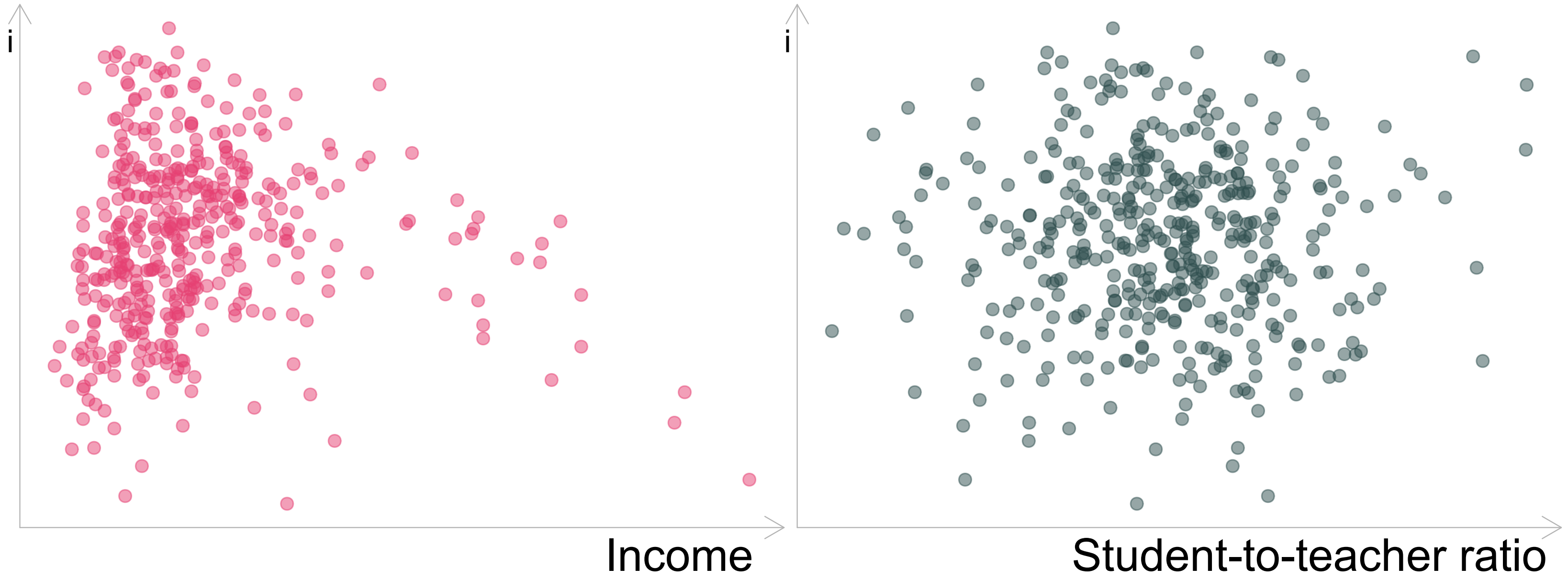
```
#> # A tibble: 3 x 5
#>   term        estimate std.error statistic   p.value
#>   <chr>          <dbl>     <dbl>     <dbl>     <dbl>
#> 1 (Intercept)  639.        7.45       85.7  5.70e-267
#> 2 ratio         -0.649     0.354      -1.83 6.79e-  2
#> 3 income         1.84      0.0928     19.8  4.38e- 62
```

# Testing for heteroskedasticity

## Examples

Now, let's see what the residuals suggest about heteroskedasticity

```
# Add the residuals to our dataset
test_df$e ← residuals(est_model)
```



# Testing for heteroskedasticity

## Example: Goldfeld-Quandt

Income looks potentially heteroskedastic; let's test via Goldfeld-Quandt.

```
# Arrange the data by income
test_df ← arrange(test_df, income)
# Re-estimate the model for the last and first 158 observations
est_model1 ← lm(test_score ~ ratio + income, data = tail(test_df, 158))
est_model2 ← lm(test_score ~ ratio + income, data = head(test_df, 158))
# Grab the residuals from each regression
e_model1 ← residuals(est_model1)
e_model2 ← residuals(est_model2)
# Calculate SSE for each regression
(sse_model1 ← sum(e_model1^2))
```

```
#> [1] 19305.01
```

```
(sse_model2 ← sum(e_model2^2))
```

```
#> [1] 29537.83
```

## Example: Goldfeld-Quandt

Remember the Goldfeld-Quandt test statistic?

$$F_{n^\star-k,\,n^\star-k} = \dfrac{\text{RSS}_2}{\text{RSS}_1} \approx \dfrac{29,537.83}{19,305.01} \approx 1.53 \quad \text{Test via } F_{158-3,\,158-3}$$

```r
# G-Q test statistic
(f_gq ← sse_model2/sse_model1)
```

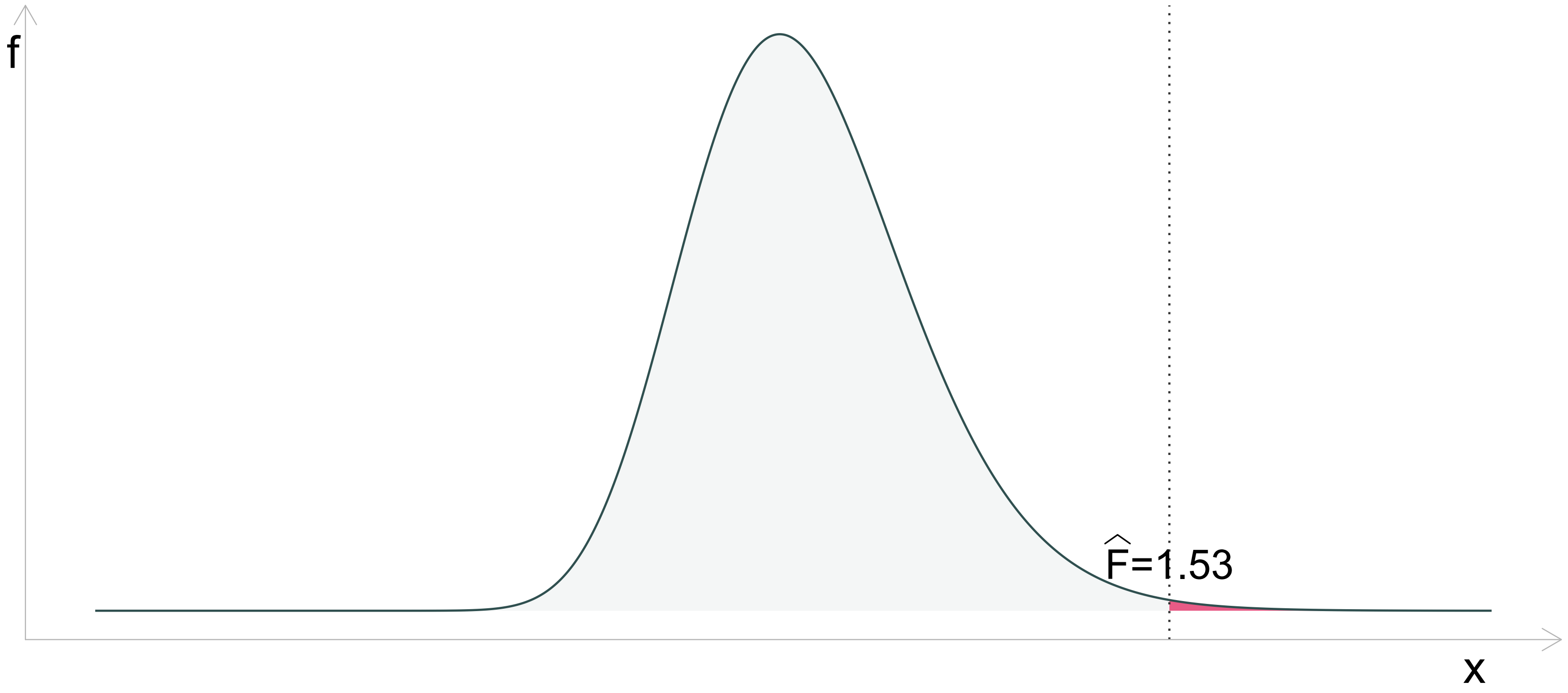
```
#> [1] 1.530061
```

```r
# p-value
pf(q = f_gq, df1 = 158-3, df2 = 158-3, lower.tail = F)
```

```
#> [1] 0.004226666
```

## Example: Goldfeld-Quandt

The Goldfeld-Quandt test statistic and its null distribution



# Testing for heteroskedasticity

## Example: Goldfeld-Quandt

Putting it all together:

$H_0$: $\sigma_1^2 = \sigma_2^2$ *vs.* $H_A$: $\sigma_1^2 \neq \sigma_2^2$

Goldfeld-Quandt test statistic: $F \approx 1.53$

*p*-value $\approx 0.00423$

$\therefore$ Reject $H_0$ (*p*-value is less than 0.05).

**Conclusion:** There is statistically significant evidence that $\sigma_1^2 \neq \sigma_2^2$. Therefore, we find statistically significant evidence of heteroskedasticity (at the 5-percent level).

# Testing for heteroskedasticity

## Example: Goldfeld-Quandt

What if we had chosen to focus on student-to-teacher ratio?

```r
# Arrange the data by ratio
test_df ← arrange(test_df, ratio)
# Re-estimate the model for the last and first 158 observations
est_model3 ← lm(test_score ~ ratio + income, data = tail(test_df, 158))
est_model4 ← lm(test_score ~ ratio + income, data = head(test_df, 158))
# Grab the residuals from each regression
e_model3 ← residuals(est_model3)
e_model4 ← residuals(est_model4)
# Calculate SSE for each regression
(sse_model3 ← sum(e_model3^2))
```

```
#> [1] 26243.52
```

```r
(sse_model4 ← sum(e_model4^2))
```

```
#> [1] 29101.52
```

## Example: Goldfeld-Quandt

$$F_{n^\star - k,\, n^\star - k} = \dfrac{\text{RSS}_4}{\text{RSS}_3} \approx \dfrac{29,101.52}{26,243.52} \approx 1.11$$

which has a *p*-value of approximately 0.2603.

$\therefore$ We would have failed to reject $H_0$, concluding that we failed to find statistically significant evidence of heteroskedasticity.

**Lesson:** Understand the limitations of estimators, tests, *etc.*

# Testing for heteroskedasticity

## Example: Breusch-Pagan

Let's test the same model with the Breusch Pagan.

*Recall:* We saved our residuals as `e` in our dataset, *i.e.*,

```
test_df$e <- residuals(est_model)
```

# Testing for heteroskedasticity

## Example: Breusch-Pagan

and use the resulting $R^2$ to calculate a test statistic.

```r
# Regress squared residuals on explanatory variables
bp_model <- lm(I(e^2) ~ ratio + income, data = test_df)
# Grab the R-squared
(bp_r2 <- summary(bp_model)$r.squared)
```

```
#> [1] 3.23205e-05
```

# Testing for heteroskedasticity

## Example: Breusch-Pagan

The Breusch-Pagan test statistic is

$$\text{LM} = n \times R_u^2 \approx 420 \times 0.0000323 \approx 0.0136$$

which we test against a $\chi_k^2$ distribution (here: $k = 2$).[†]

```r
# B-P test statistic
bp_stat <- 420 * bp_r2
# Calculate the p-value
pchisq(q = bp_stat, df = 2, lower.tail = F)
```

```
#> [1] 0.9932357
```

[†]: $k$ is the number of explanatory variables (excluding the intercept).

# Testing for heteroskedasticity

## Example: Breusch-Pagan

H$_0$: $\alpha_1 = \alpha_2 = 0$ *vs.* H$_A$: $\alpha_1 \neq 0$ and/or $\alpha_2 \neq 0$

for the model $u_i^2 = \alpha_0 + \alpha_1 \text{Ratio}_i + \alpha_2 \text{Income}_i + w_i$

Breusch-Pagan test statistic: $\widehat{\text{LM}} \approx 0.014$

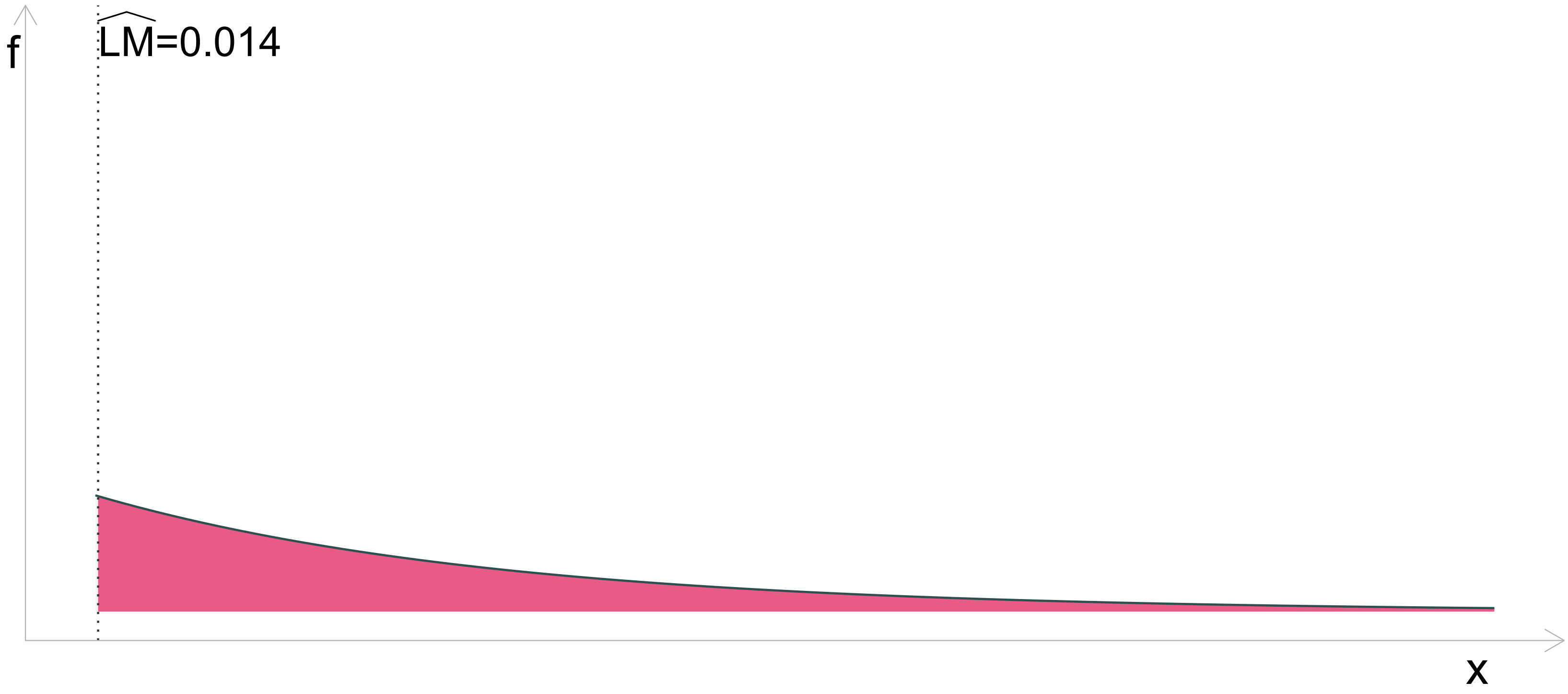
*p*-value $\approx 0.993$

$\therefore$ Fail to reject H$_0$ (the *p*-value is greater than 0.05)

**Conclusion:** We do not find statistically significant evidence of heteroskedasticity at the 5-percent level. (We find no evidence of a *linear* relationship between $u_i^2$ and the explanatory variables.)

## Example: Breusch-Pagan

The Breusch-Pagan test statistic and its null distribution



## Example: White

The **White test** adds squared terms and interactions to the **B-P test**.

$$\begin{aligned}u_i^2 =&\alpha_0 + \alpha_1 \text{Ratio}_i + \alpha_2 \text{Income}_i \\ &+ \alpha_3 \text{Ratio}_i^2 + \alpha_4 \text{Income}_i^2 + \alpha_5 \text{Ratio}_i \times \text{Income}_i \\ &+ w_i\end{aligned}$$

which moves the null hypothesis from

$H_0$: $\alpha_1 = \alpha_2 = 0$ to

$H_0$: $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 0$

So we just need to update our R code, and we're set.

## Example: White

*Aside:* R has funky notation for squared terms and interactions in `lm()`:

- **Squared terms** use `I()`, *e.g.*, `lm(y ~ I(x^2))`
- **Interactions** use `:` between the variables, *e.g.*, `lm(y ~ x1:x2)`

*Example:* Regress `y` on quadratic of `x1` and `x2`:

```
# Pretend quadratic regression w/ interactions
lm(y ~ x1 + x2 + I(x1^2) + I(x2^2) + x1:x2, data = pretend_df)
```

## Example: White

**Step 4:** Calculate the associated *p*-value (where $\text{LM} \overset{d}{\sim} \chi_k^2$); here, $k = 5$

```
# Regress squared residuals on quadratic of explanatory variables
white_model <- lm(
  I(e^2) ~ ratio + income + I(ratio^2) + I(income^2) + ratio:income,
  data = test_df
)
# Grab the R-squared
white_r2 <- summary(white_model)$r.squared
# Calculate the White test statistic
white_stat <- 420 * white_r2
# Calculate the p-value
pchisq(q = white_stat, df = 5, lower.tail = F)
```

```
#> [1] 1.028039e-05
```

# Heteroskedasticity

## Example: White

Putting everything together...

$H_0$: $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 0$ *vs.* $H_A$: $\alpha_i \neq 0$ for some $i \in \{1, 2, \ldots, 5\}$

$$\begin{aligned} u_i^2 =& \alpha_0 + \alpha_1 \text{Ratio}_i + \alpha_2 \text{Income}_i \\ & + \alpha_3 \text{Ratio}_i^2 + \alpha_4 \text{Income}_i^2 \\ & + \alpha_5 \text{Ratio}_i \times \text{Income}_i + w_i \end{aligned}$$

Our White test statistic: $\text{LM} = n \times R_e^2 \approx 420 \times 0.073 \approx 30.8$

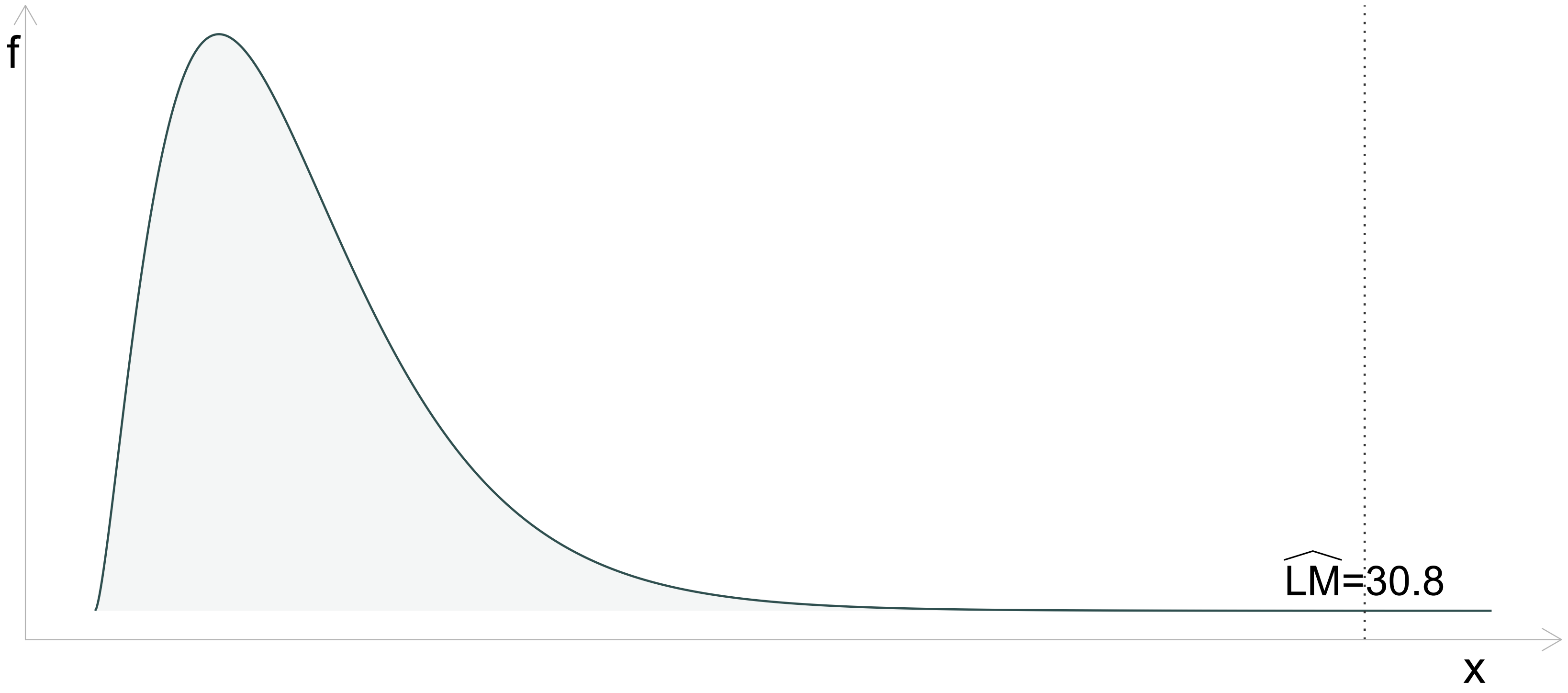
Under the $\chi_5^2$ distribution, this $\widehat{\text{LM}}$ has a *p*-value less than 0.001.

$\therefore$ We **reject $H_0$** and conclude there is **statistically significant evidence of heteroskedasticity** (at the 5-percent level).

# Heteroskedasticity

## Example: White

The White test statistic and its null distribution



# Heteroskedasticity

## Review questions

- **Q:** What is the definition of heteroskedasticity?
- **Q:** Why are we concerned about heteroskedasticity?
- **Q:** Does plotting $y$ against $x$, tell us anything about heteroskedasticity?
- **Q:** Does plotting $\hat{u}_i$ against $x$, tell us anything about heteroskedasticity?
- **Q:** Since we cannot observe the $u_i$'s, what do we use to *learn about* heteroskedasticity?
- **Q:** Which test do you recommend to test for heteroskedasticity? Why?

# Heteroskedasticity

## Remedies

1. Ensure your specification doesn't cause heteroskedasticity
2. Increase efficiency by weighting our observations
3. Ignore OLS's inefficiency, focus on unbiased estimates for our standard errors which leads to correct inference

**For details**, see Chapter 7 of Dougherty. Future metrics classes such as EC421 will go into the weeds on this matter.