Statistics Review I

EC 320: Introduction to Econometrics

Kyle Raze

Fall 2019

Prologue

Housekeeping

I'll post Problem Set 1 tonight.

- Due next Monday on Canvas.
- Analytical and computational components (turned in separately).

Issues with R?

- Both Saurabh and I have office hours tomorrow (12:00-14:00 and 15:00-16:00).

Motivation

The focus of our course is **regression analysis**, a useful toolkit for learning from data.

To understand regression, its mechanics, and its pitfalls, **we need to understand the underlying statistical theory.**

- Insights from theory can help us become better practitioners and savvier consumers of science.

Today, we will review important concepts you learned in Math 243.

- Maybe some you missed, too.

A Brief Math Review

Notation

Data on a variable X **are**[*] a sequence of n observations, indexed by i:

$$\{x_i : 1, \dots, n\}.$$

Example: $n = 5$

i	x_i
1	4
2	2
3	8
4	10
5	6

- i indicates the row number.
- n is the number of rows.
- x_i is the value of X for row i.

[*] *Data* = **plural** of *datum*.

Summation

The **summation operator** adds a sequence of numbers over an index:

$$\sum_{i=1}^{n} x_i \equiv x_1 + x_2 + \cdots + x_n.$$

- "The sum of x_i from 1 to n."

Example: $n = 4$

i	x_i
1	7
2	4
3	10
4	2

$$\begin{aligned}\sum_{i=1}^{4} x_i &= 7 + 4 + 10 + 2 \\ &= 23\end{aligned}$$

Summation

Rule 1

For any constant c,

$$\sum_{i=1}^{n} c = nc.$$

Example: $n = 4$

i	c
1	2
2	2
3	2
4	2

$$\begin{aligned}\sum_{i=1}^{4} 2 &= 4 \times 2 \\ &= 8\end{aligned}$$

Summation

Rule 2

For any constant c,

$$\sum_{i=1}^{n} cx_i = c\sum_{i=1}^{n} x_i.$$

Example: $n = 3$

i	c	x_i
1	2	7
2	2	4
3	2	10

$$\begin{aligned}\sum_{i=1}^{3} 2x_i &= 2 \times 7 + 2 \times 4 + 2 \times 10 \\ &= 14 + 8 + 20 \\ &= 42\end{aligned}$$

$$\begin{aligned}2\sum_{i=1}^{3} x_i &= 2(7 + 4 + 10) \\ &= 42\end{aligned}$$

Summation

Rule 3

If $\{(x_i, y_i) : 1, \ldots, n\}$ is a set of n pairs, and a and b are constants, then

$$\sum_{i=1}^{n}(ax_i + by_i) = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} y_i.$$

Example: $n = 2$

i	a	x_i	b	y_i
1	2	7	1	4
2	2	4	1	2

$$\begin{aligned}\sum_{i=1}^{2}(2x_i + y_i) &= 18 + 10 \\ &= 28\end{aligned}$$

$$\begin{aligned}2\sum_{i=1}^{2} x_i + \sum_{i=1}^{2} y_i &= 2 \times 11 + 6 \\ &= 28\end{aligned}$$

Caution

The **sum of the ratios** **is not** the **ratio of the sums**:

$$\sum_{i=1}^{n} x_i/y_i \neq \left(\sum_{i=1}^{n} x_i\right) \bigg/ \left(\sum_{i=1}^{n} y_i\right).$$

- If $n = 2$, then $\frac{x_1}{y_1} + \frac{x_2}{y_2} \neq \frac{x_1+x_2}{y_1+y_2}$.

The **sum of squares** **is not** the **square of the sums**:

$$\sum_{i=1}^{n} x_i^2 \neq \left(\sum_{i=1}^{n} x_i\right)^2.$$

- If $n = 2$, then $x_1^2 + x_2^2 \neq (x_1 + x_2)^2 = x_1^2 + 2x_1x_2 + x_2^2$.

Probability Review

Random Variables

Experiment: Any procedure that is *infinitely repeatable* and has a *well-defined set of outcomes*.

- Flip a coin 10 times and record the number of heads.
- Roll two six-sided dice and record the sum.

Random Variable: A variable with *numerical values determined by an experiment or a random phenomenon*.

- Describes the sample space of an experiment.
- **Sample space:** The set of potential outcomes an experiment could generate, *e.g.*, the sum of two dice is an integer from 2 to 12.
- **Event:** A subset of the sample space or a combination of outcomes, *e.g.*, rolling a two or a four.

Random Variables

Notation: capital letters for random variables (*e.g.*, X, Y, or Z) and lowercase letters for particular outcomes (*e.g.*, x, y, or z).

Example 1: Flipping a coin.

- Two outcomes: heads or tails.
- Quantify the outcomes: Define a random variable $\textbf{Heads}$ such that $\textbf{Heads} = \mathbf{1}$ if heads and $\textbf{Heads} = \mathbf{0}$ if tails.

Example 2: Flipping a coin 10 times.

- Several outcomes: 10 heads and 0 tails, 9 heads and 1 tails, 8 heads and 2 tails, *etc*.
- The number of heads is a random variable: $\{\textbf{Heads} : \mathbf{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}\}$.

Discrete Random Variables

Discrete Random Variable: A random variable that takes a countable set of values.

A **Bernoulli** (or binary) random variable takes values of either 1 or 0.

- Characterized by $\mathbb{P}(X = 1)$, "the probability of success."
- Probabilities sum to 1: $\mathbb{P}(X = 1) + \mathbb{P}(X = 0) = 1$.
    - For a "fair" coin, $\mathbb{P}(\textbf{Heads} = 1) = \frac{1}{2} \implies \mathbb{P}(\textbf{Heads} = 0) = \frac{1}{2}$.
- More generally, if $\mathbb{P}(X = 1) = \theta$ for some $\theta \in [0, 1]$, then $\mathbb{P}(X = 0) = 1 - \theta$.
    - If the probability of passing this class is 75%, then the probability of not passing is 25%.

Discrete Random Variables

Probabilities

We describe a discrete random variable by listing its possible values with associated probabilities.

If X takes on k possible values $\{x_1, \ldots, x_k\}$, then the probabilities $p_1, p_2, \ldots, p_k$ are defined by

$$p_j = \mathbb{P}(X = x_j), \quad j = 1, 2, \ldots, k,$$

where

$$p_j \in [0, 1]$$

and

$$p_1 + p_2 + \cdots + p_k = 1.$$

Discrete Random Variables

Probability density function

The **probability density function** (**pdf**) of X summarizes possible outcomes and associated probabilities:

$$f(x_j) = p_j, \quad j = 1, 2, \ldots, k.$$

Example

2020 Presidential election: 538 electoral votes at stake.

- $\{X : 0, 1, \ldots, 538\}$ is the number of electoral votes won by the Democratic candidate.
- Extremely unlikely that she will win 0 votes or all 538 votes: $f(0) \approx 0$ and $f(538) \approx 0$.
- Nonzero probability of winning an exact majority: $f(270) > 0$.

Discrete Random Variables

Example

Basketball player goes to the foul line to shoot two free throws.

- X is the number of shots made (either 0, 1, or 2).
- The pdf of X is $f(0) = 0.3$, $f(1) = 0.4$, $f(2) = 0.3$.
- **Note:** the probabilities sum to 1.

Use the pdf to calculate the probability of the **event** that the player makes *at least one shot, i.e.,* $\mathbb{P}(X \geq 1)$.

- $\mathbb{P}(X \geq 1) = \mathbb{P}(X = 1) + \mathbb{P}(X = 2) = 0.4 + 0.3 = 0.7$.

Continuous Random Variables

Continuous Random Variable: A random variable that takes any real value with *zero* probability.

- **Wait, what?!** The variable takes so many values that we can't count all possibilities, so the probability of any one particular value is zero.

Measurement is discrete (*e.g.*, dollars and cents), but variables with many possible values are best treated as continuous.

- *e.g.*, electoral votes, height, wages, temperature, *etc.*

Continuous Random Variables

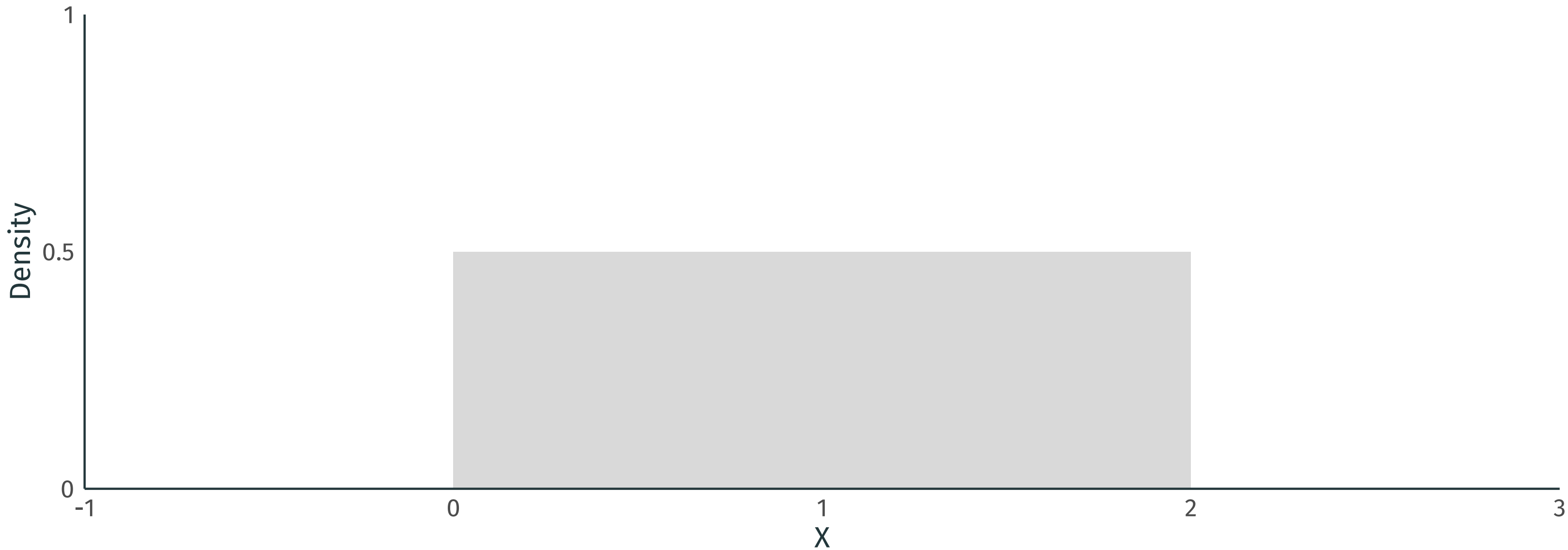
Probability density functions also describe continuous random variables.

- Difference: Interested in the probability of events within a *range* of values.
- *e.g.* What is the probability of more than 1 inch of rain tomorrow?

Uniform Distribution

The probability density function of a variable uniformly distributed between 0 and 2 is

$$f(x) = \begin{cases} \frac{1}{2} & \text{if } 0 \leq x \leq 2 \\ 0 & \text{if } x < 0 \text{ or } x > 2 \end{cases}$$



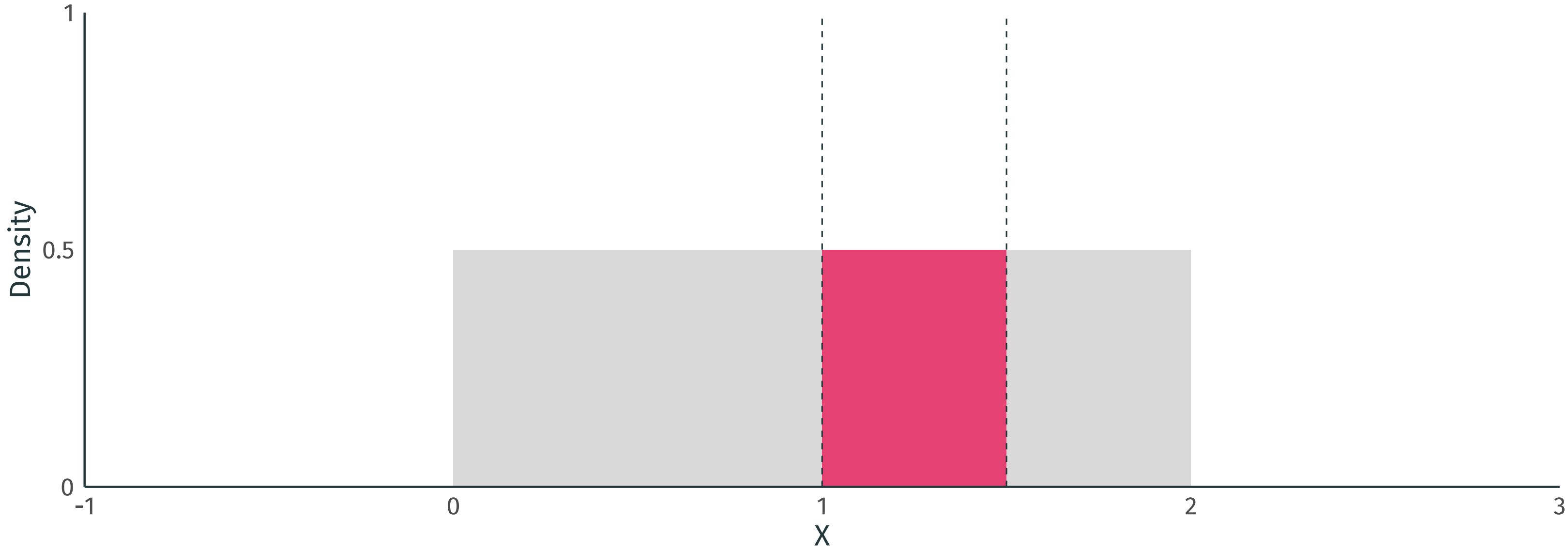
Continuous Random Variables

Uniform Distribution

By definition, the area under $f(x)$ is equal to 1.

The **shaded area** illustrates the probability of the event $1 \leq X \leq 1.5$.

- $\mathbb{P}(1 \leq X \leq 1.5) = (1.5 - 1) \times 0.5 = 0.25$.

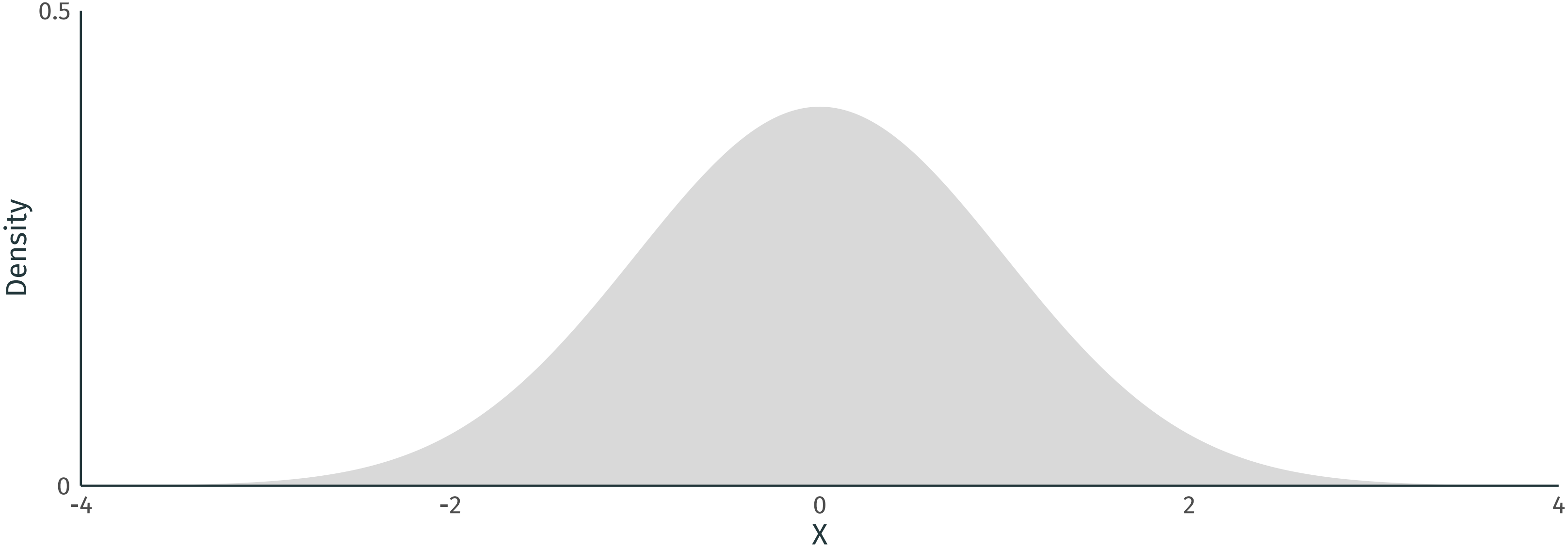


Continuous Random Variables

Normal Distribution

The "bell curve."

- Symmetric: mean and median occur at the same point (*i.e.*, no skew).
- Low-probability events in tails; high-probability events near center.

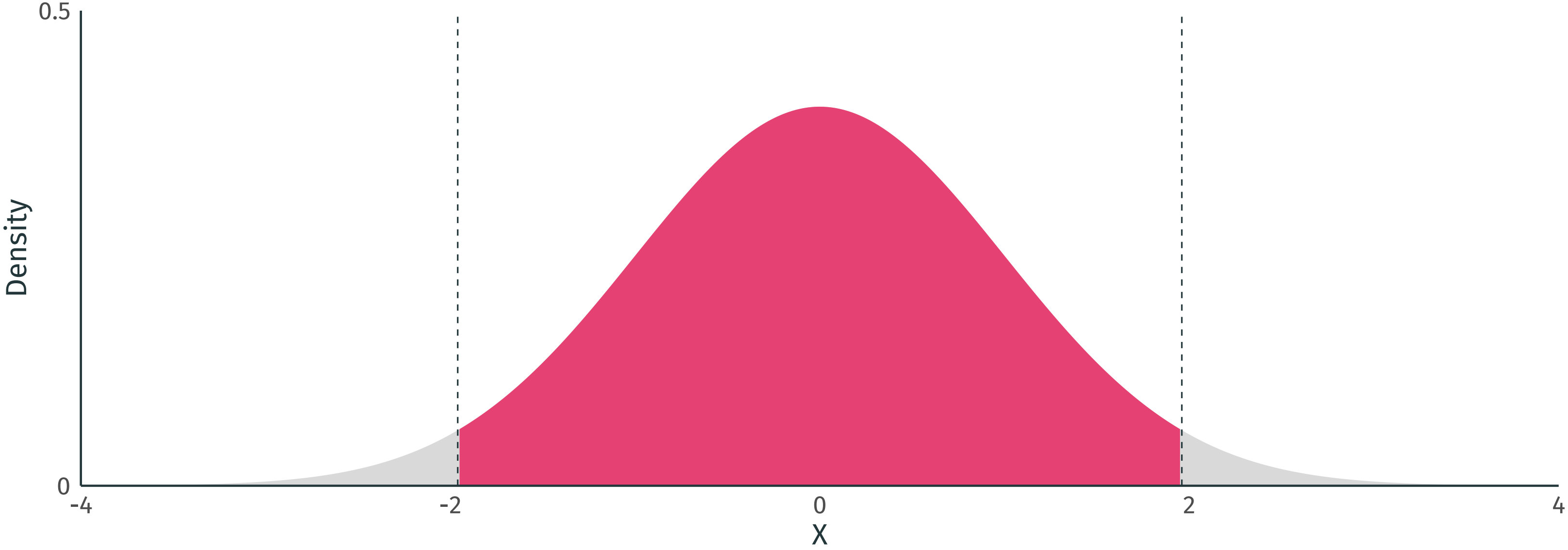


Continuous Random Variables

Normal Distribution

The **shaded area** illustrates the probability of the event $-\mathbf{2} \leq \boldsymbol{X} \leq \mathbf{2}$.

- "Find area under curve" = use integral calculus (or, in practice, R).
- $\mathbb{P}(-\mathbf{2} \leq \boldsymbol{X} \leq \mathbf{2}) \approx \mathbf{0.95}$.



Expected Value

A density function describes an entire distribution, but sometimes we just want a summary.

The **expected value** describes the *central tendency* of distribution in a single number.

- *Central tendency* = typical value.

Expected Value

Definition (Discrete)

The expected value of a discrete random variable X is the weighted average of its k values $\{x_1, \ldots, x_k\}$ and their associated probabilities:

$$\begin{aligned}\mathbb{E}(X) &= x_1\,\mathbb{P}(x_1) + x_2\,\mathbb{P}(x_2) + \cdots + x_k\,\mathbb{P}(x_k)\\ &= \sum_{j=1}^{k} x_j\,\mathbb{P}(x_j).\end{aligned}$$

- Also known as the **population mean**.

Expected Value

Example

Rolling a six-sided die once can take values $\{1, 2, 3, 4, 5, 6\}$, each with equal probability. **What is the expected value of a roll?**

$$\mathbb{E}(\text{Roll}) = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 6 \times \frac{1}{6} = 3.5.$$

- **Note:** The expected value can be a number that isn't a possible outcome of X.

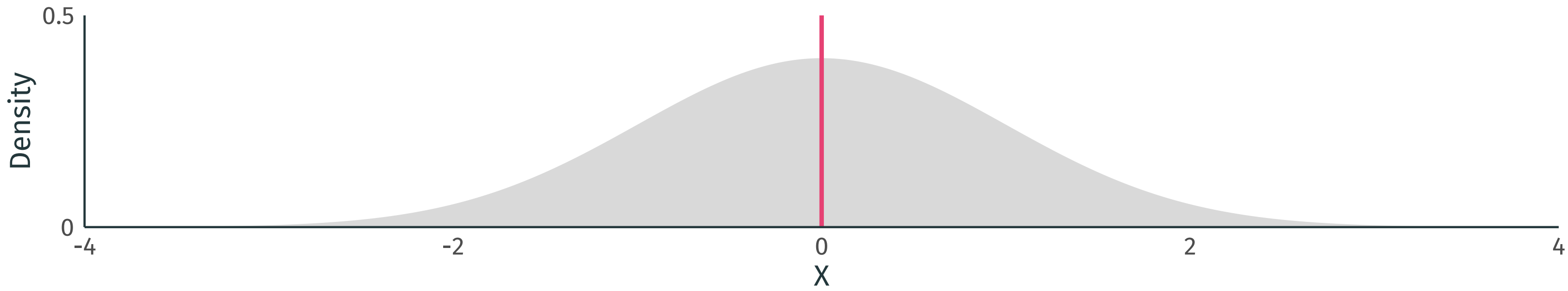
Expected Value

Definition (Continuous)

If X is a continuous random variable and $f(x)$ is its probability density function, then the expected value of X is

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x f(x) dx.$$

- **Note:** x represents the particular values of X.
- Same idea as the discrete definition: describes the **population mean**.



Expected Value

Rule 1

For any constant c, $\mathbb{E}(c) = c$.

Not-so-exciting examples

$\mathbb{E}(5) = 5$.

$\mathbb{E}(1) = 1$.

$\mathbb{E}(4700) = 4700$.

Expected Value

Rule 2

For any constants a and b, $\mathbb{E}(aX + b) = a\,\mathbb{E}(X) + b$.

Example

Suppose X is the high temperature in degrees Celsius in Eugene during August. The long-run average is $\mathbb{E}(X) = 28$. If Y is the temperature in degrees Fahrenheit, then $Y = 32 + \frac{9}{5}X$. **What is $\mathbb{E}(Y)$?**

- $\mathbb{E}(Y) = 32 + \frac{9}{5}\,\mathbb{E}(X) = 32 + \frac{9}{5} \times 28 = 82.4$.

Expected Value

Rule 3

If $\{a_1, a_2, \ldots, a_n\}$ are constants and $\{X_1, X_2, \ldots, X_n\}$ are random variables, then

$$\mathbb{E}(a_1 X_1 + a_2 X_2 + \cdots + a_n X_n) = a_1 \, \mathbb{E}(X_1) + a_2 \, \mathbb{E}(X_2) + \cdots + a_n \, \mathbb{E}(X_n).$$

In English, **the expected value of the sum** = **the sum of expected values**.

Expected Value

Rule 3

The expected value of the sum = **the sum of expected values**.

Example

Suppose that a coffee shop sells X_1 small, X_2 medium, and X_3 large caffeinated beverages in a day. The quantities sold are random with expected values $\mathbb{E}(X_1) = 43$, $\mathbb{E}(X_2) = 56$, and $\mathbb{E}(X_3) = 21$. The prices of small, medium, and large beverages are 1.75, 2.50, and 3.25 dollars. **What is expected revenue?**

$$
\begin{aligned}
\mathbb{E}(1.75X_1 + 2.50X_2 + 3.35X_n) &= 1.75\,\mathbb{E}(X_1) + 2.50\,\mathbb{E}(X_2) + 3.25\,\mathbb{E}(X_3) \\
&= 1.75(43) + 2.50(56) + 3.25(21) \\
&= 283.5
\end{aligned}
$$

Caution

Previously, we found that the expected value of rolling a six-sided die is $\mathbb{E}(\text{Roll}) = 3.5$.

- If we square this number, we get $\left[\mathbb{E}(\text{Roll})\right]^2 = 12.25$.

Is $\left[\mathbb{E}(\text{Roll})\right]^2$ the same as $\mathbb{E}\left(\text{Roll}^2\right)$?

No!

$$\begin{aligned}
\mathbb{E}\left(\text{Roll}^2\right) &= 1^2 \times \frac{1}{6} + 2^2 \times \frac{1}{6} + 3^2 \times \frac{1}{6} + 4^2 \times \frac{1}{6} + 5^2 \times \frac{1}{6} + 6^2 \times \frac{1}{6} \\
&\approx 15.167 \\
&\neq 12.25.
\end{aligned}$$

Caution

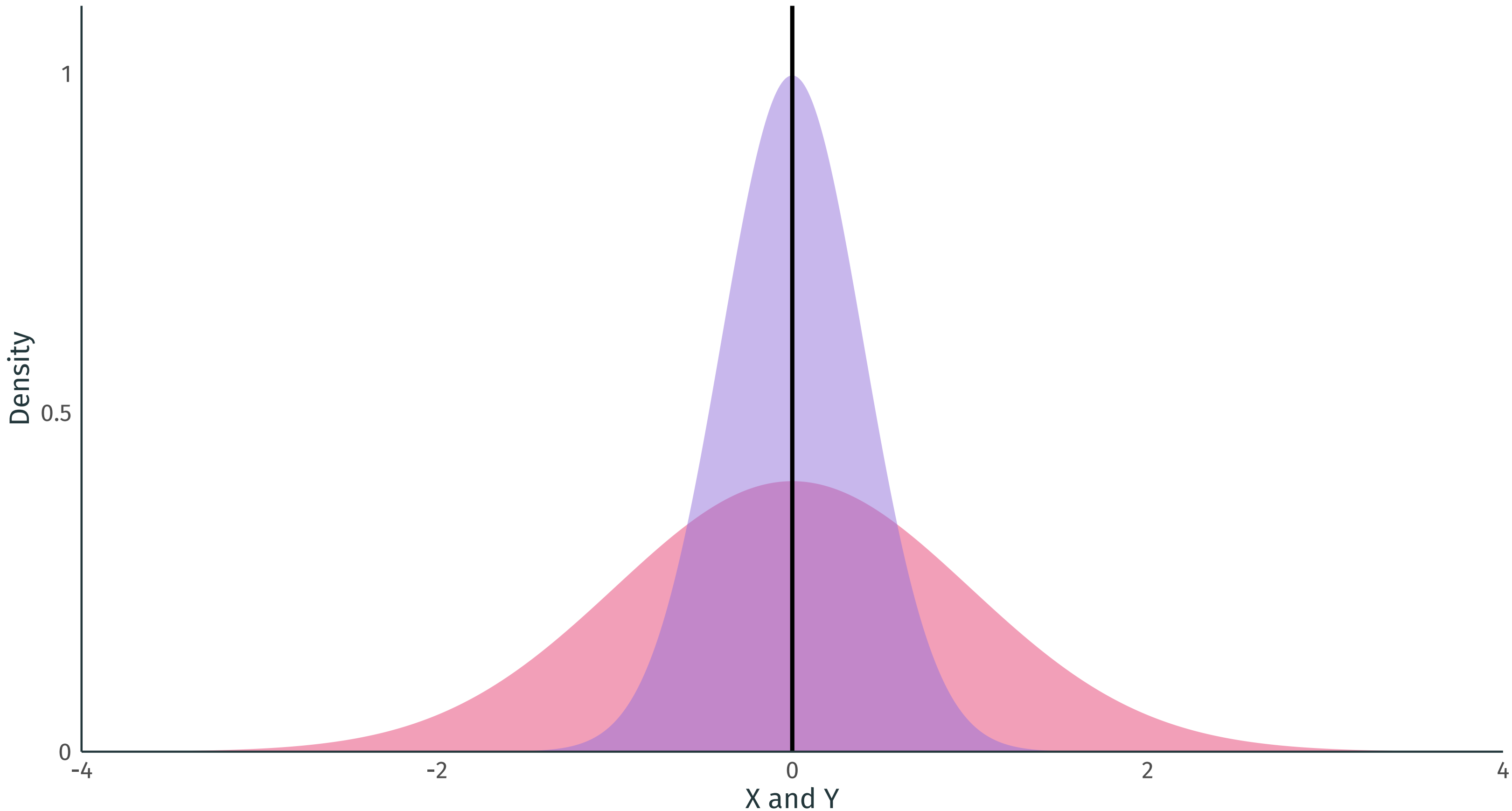
Except in special cases, **the transformation of an expected value** **is not** **the expected value of a transformed random variable**.

For some function $g(\cdot)$, it is typically the case that

$$g\left(\mathbb{E}(X)\right) \neq \mathbb{E}(g(X)).$$

Variance

Random variables X and Y share the same population mean, but are distributed differently.



Variance

How tightly is a random variable distributed about its mean?

- Let $\mu = \mathbb{E}(X)$.
- Describe the distance of X from its population mean μ as the squared difference: $(X - \mu)^2$.

Variance tells us how far X deviates from μ, *on average*:

$$\mathbf{Var}(X) \equiv \mathbb{E}\big((X - \mu)^2\big) = \sigma^2$$

- σ^2 is shorthand for variance.

Variance

Rule 1

$\mathbf{Var}(X) = \mathbf{0} \iff X$ is a constant.

- If a random variable never deviates from its mean, then it has zero variance.
- If a random variable is always equal to its mean, then it's a (not-so-random) constant.

Variance

Rule 2

For any constants a and b, $\text{Var}(aX + b) = a^2 \,\text{Var}(X)$.

Example

Suppose X is the high temperature in degrees Celsius in Eugene during August. If Y is the temperature in degrees Fahrenheit, then $Y = 32 + \frac{9}{5}X$.
What is $\text{Var}(Y)$?

- $\text{Var}(Y) = (\frac{9}{5})^2 \,\text{Var}(X) = \frac{81}{25}\,\text{Var}(X)$.

Standard Deviation

Standard deviation is the positive square root of the variance:

$$\mathrm{sd}(X) = +\sqrt{\mathrm{Var}(X)} = \sigma$$

- σ is shorthand for standard deviation.

Standard Deviation

Rule 1

For any constant c, $\mathrm{sd}(c) = 0$.

Rule 2

For any constants a and b, $\mathrm{sd}(aX + b) = |a|\,\mathrm{sd}(X)$.

Standardizing a Random Variable

When we're working with a random variable X with an unfamiliar scale, it is useful to **standardize** it by defining a new variable Z:

$$Z \equiv \frac{X - \mu}{\sigma}.$$

Z has mean 0 and standard deviation 1. How?

- First, some simple trickery: $Z = aX + b$, where $a \equiv \frac{1}{\sigma}$ and $b \equiv -\frac{\mu}{\sigma}$.
- $\mathbb{E}(Z) = a\,\mathbb{E}(X) + b = \mu\frac{1}{\sigma} - \frac{\mu}{\sigma} = 0$.
- $\mathrm{Var}(Z) = a^2\mathrm{Var}(X) = \frac{1}{\sigma^2}\sigma^2 = 1$.

Covariance

Idea: Characterize the relationship between two random variables X and Y.

Definition: $\text{Cov}(X, Y) \equiv \mathbb{E}[(X - \mu_X)(Y - \mu_Y)] = \sigma_{xy}$.

- **Positive correlation:** When $\sigma_{xy} > 0$, then X is above its mean when Y is above its mean, *on average*.

- **Negative correlation:** When $\sigma_{xy} < 0$, then X is below its mean when Y is above its mean, *on average*.

Covariance

Rule 1

If X and Y are independent, then $\mathrm{Cov}(X, Y) = 0$.

- **Statistical independence:** If X and Y are independent, then $\mathbb{E}(XY) = \mathbb{E}(X)\,\mathbb{E}(Y)$.
- $\mathrm{Cov}(X, Y) = 0$ means that X and Y are *uncorrelated*.

Caution: $\mathrm{Cov}(X, Y) = 0$ **does not imply** that X and Y are independent.

Covariance

Rule 2

For any constants a, b, c, and d, $\mathrm{Cov}(aX + b, cY + d) = ac\,\mathrm{Cov}(X, Y)$

Correlation Coefficient

A problem with covariance is that it is sensitive to units of measurement.

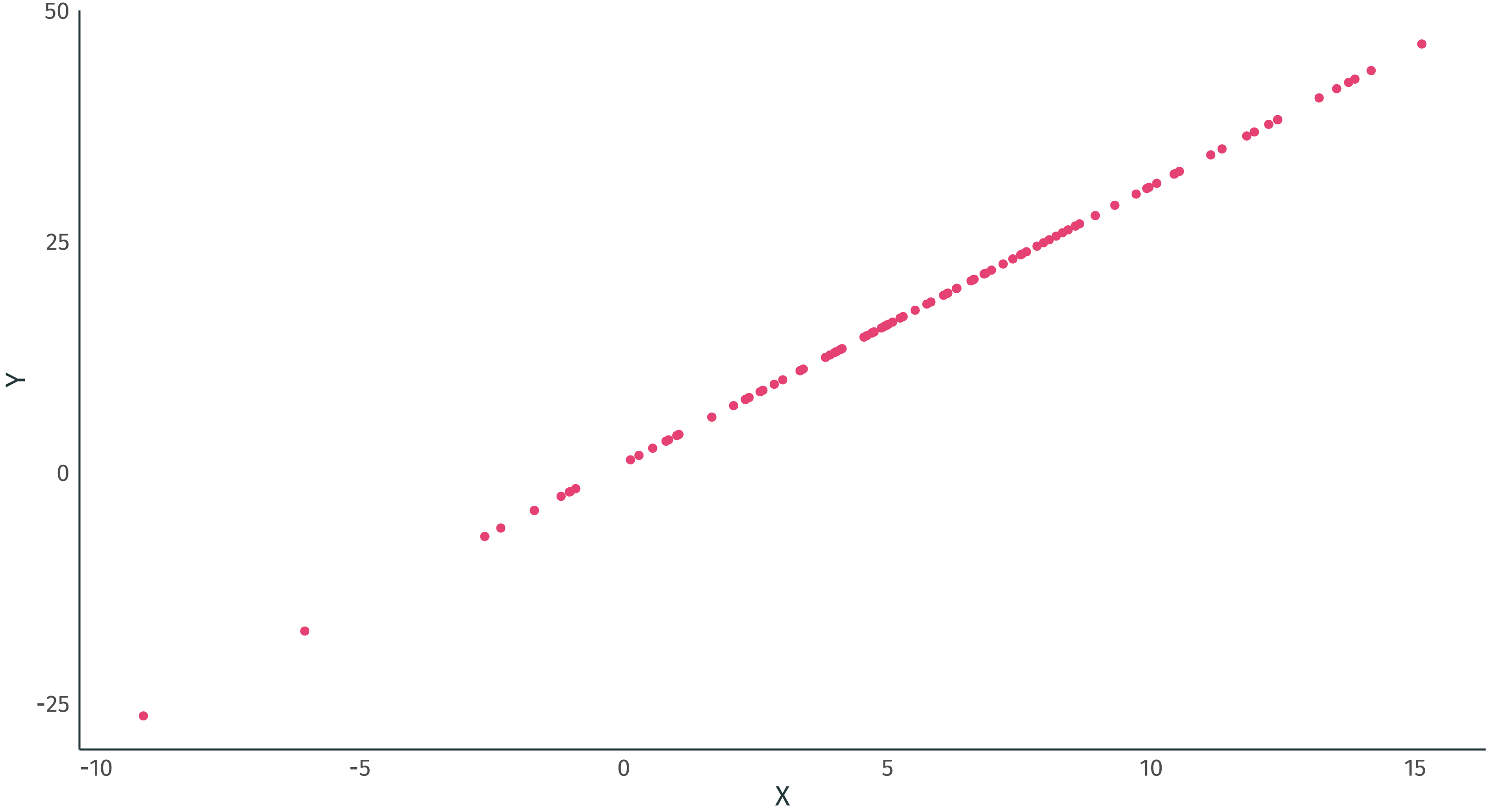
The **correlation coefficient** solves this problem by rescaling the covariance:

$$\text{Corr}(X, Y) \equiv \frac{\text{Cov}(X, Y)}{\text{sd}(X) \times \text{sd}(Y)} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}.$$

- Also denoted as ρ_{XY}.
- $-1 \leq \text{Corr}(X, Y) \leq 1$
- Invariant to scale: if I double Y, $\text{Corr}(X, Y)$ will not change.

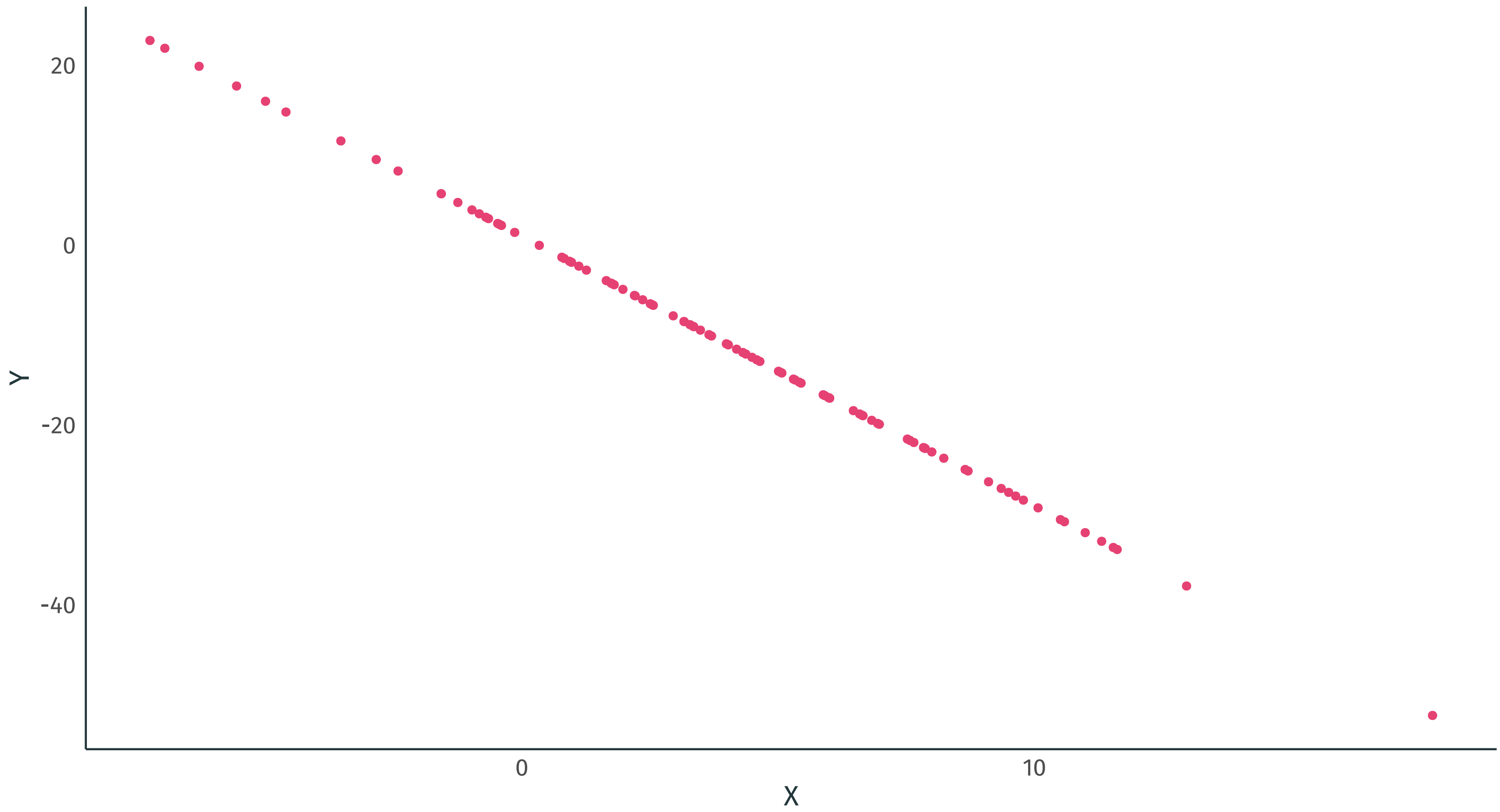
Correlation Coefficient

Perfect positive correlation: $\textbf{Corr}(X, Y) = 1$.



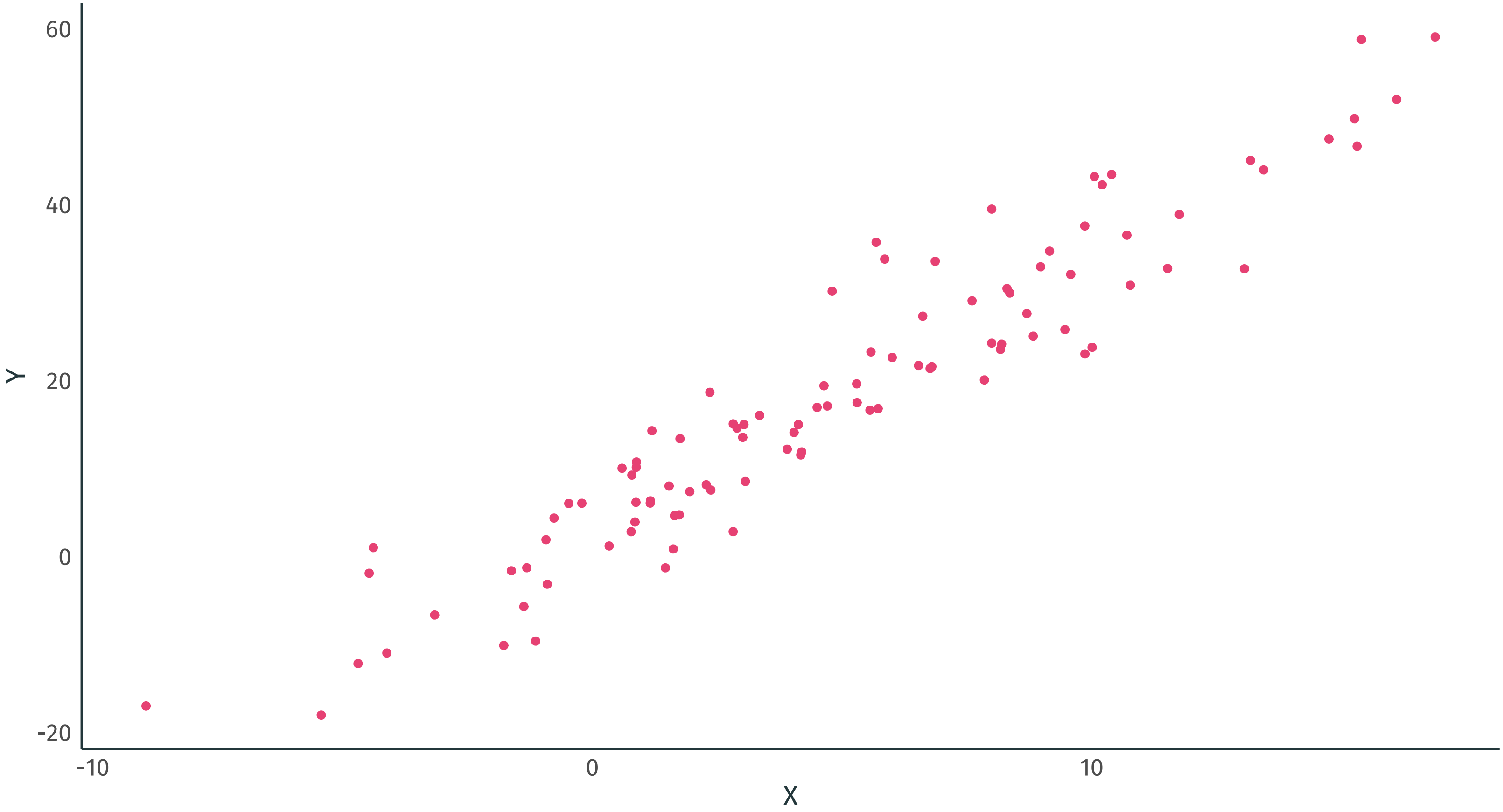
Correlation Coefficient

Perfect negative correlation: $\text{Corr}(X, Y) = -1$.



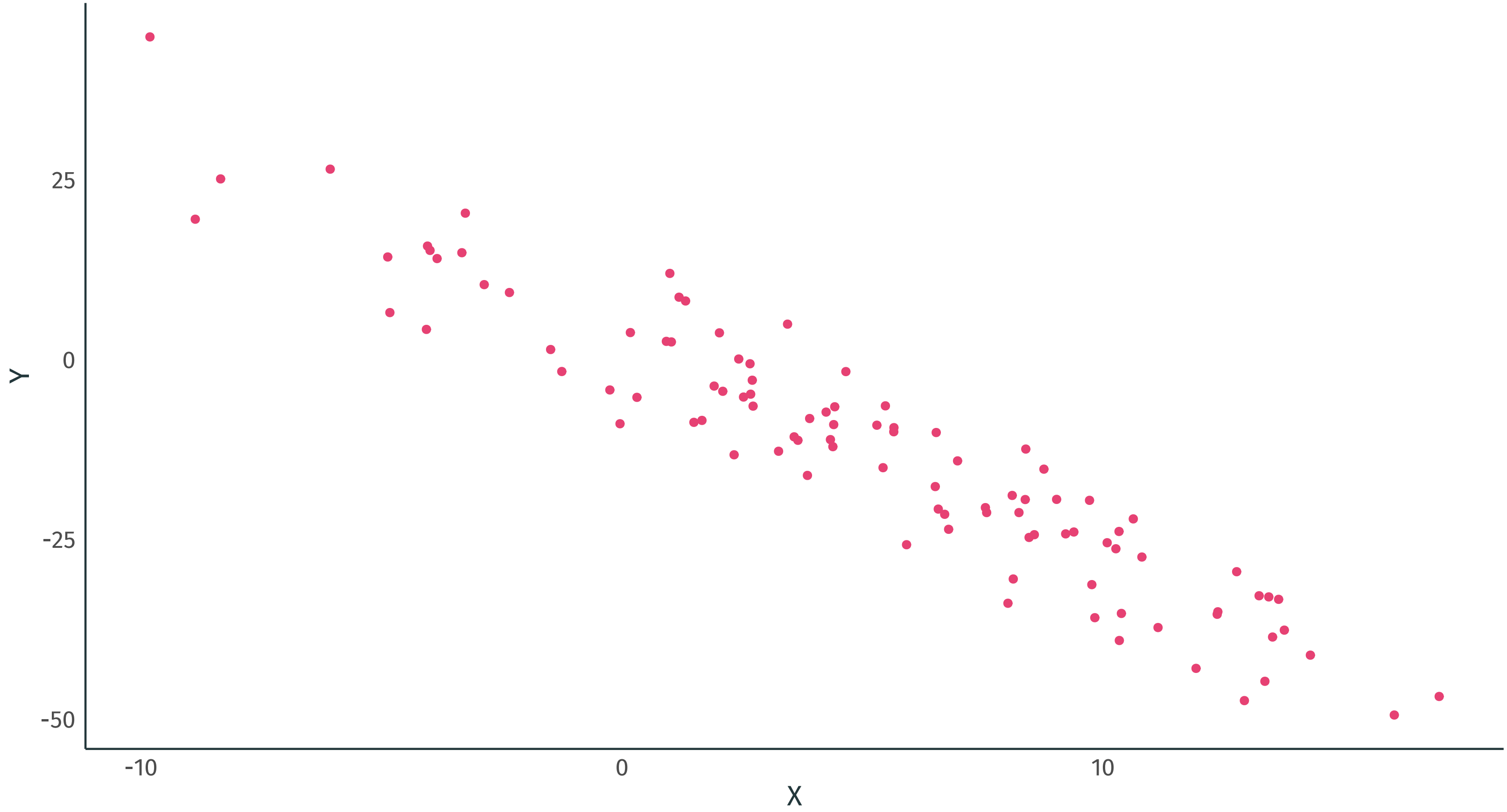
Correlation Coefficient

Positive correlation: $\mathbf{Corr}(X, Y) > \mathbf{0}$.



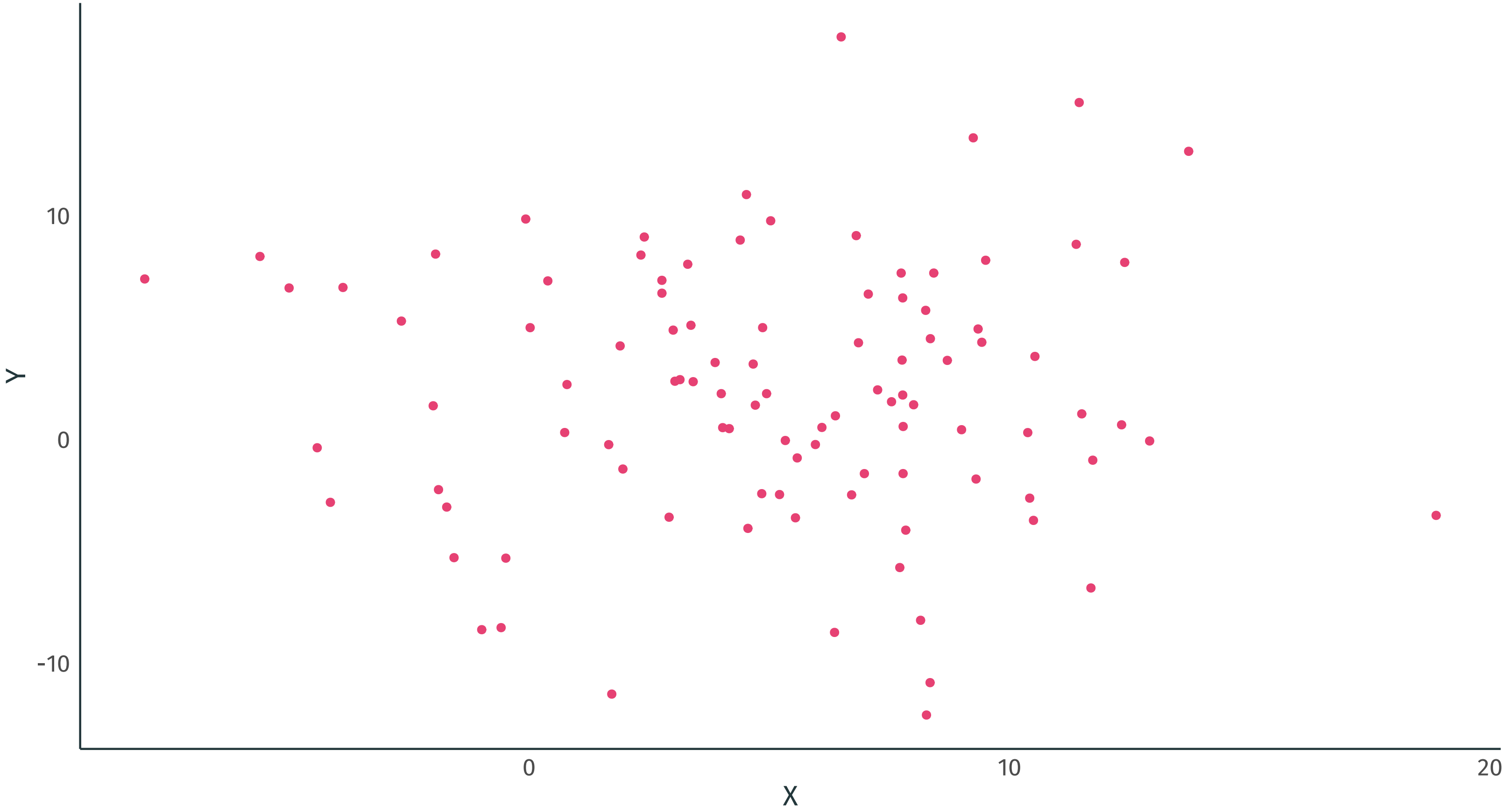
Correlation Coefficient

Negative correlation: $\mathrm{Corr}(X, Y) < 0$.



Correlation Coefficient

No correlation: $\mathbf{Corr}(X, Y) = \mathbf{0}$.



Variance, Revisited

Variance Rule 3

For constants a and b,

$$\mathrm{Var}(aX + bY) = a^2 \,\mathrm{Var}(X) + b^2 \,\mathrm{Var}(Y) + 2ab \,\mathrm{Cov}(X, Y).$$

- If X and Y are uncorrelated, then $\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$
- If X and Y are uncorrelated, then $\mathrm{Var}(X - Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$