ECON 3818

Chapter 16

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Chapter 16: Confidence Intervals

Statistical Inference

Recall, we're interested in estimating some unknown population parameter θ using the sample $X_1, \ldots, X_n$

We can use some estimator $\hat{\theta}$

- We can find its bias and its variance
- Can say what it converges to using Law of Large Numbers and the Central Limit Theorem

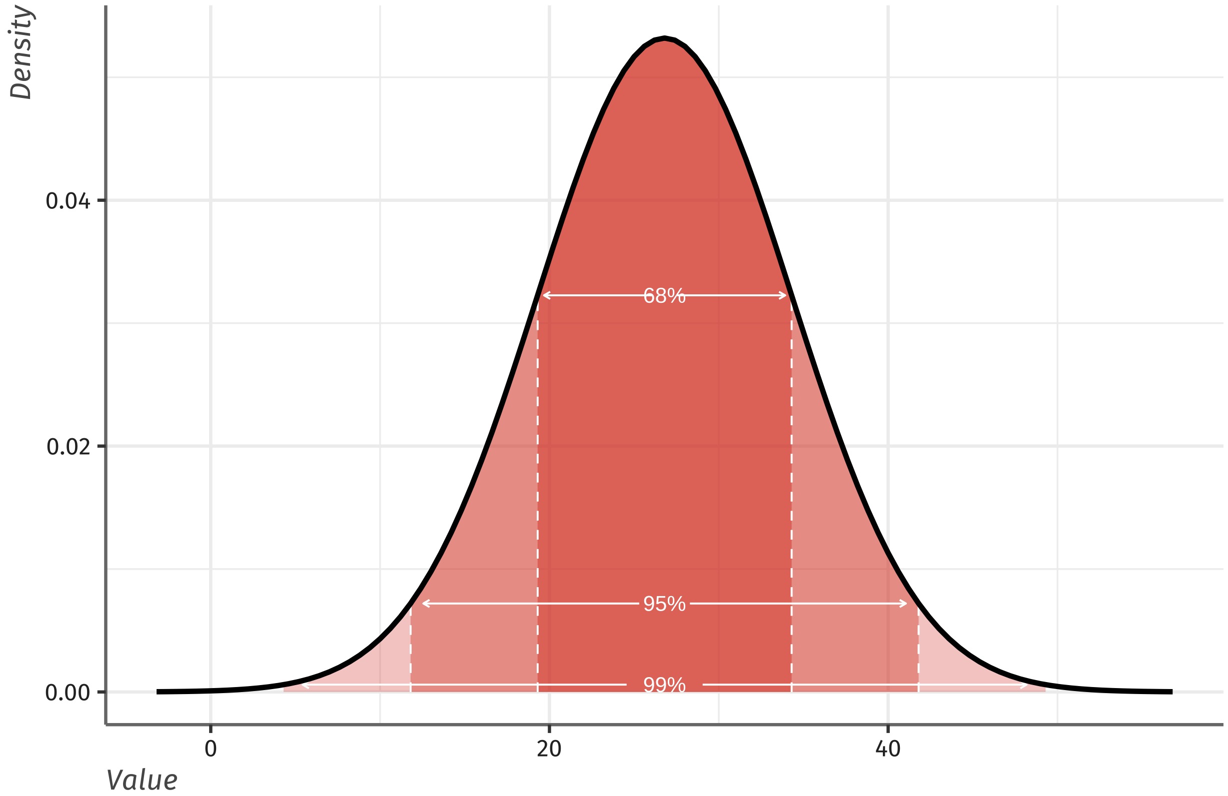
However, we don't have $n \to \infty$. We have a **finite** sample.

- Therefore, we want to construct some belief about how good our estimator is.
- For example, if we have a sample mean with 5 individuals, our sampling distribution has a large variance. We want to report that.

Example

Say we collect ACT Scores for 654 students, and calculate $\bar{X}_{654} = 26.8$. Somehow we also know that the standard deviation of the *population distribution* is σ = 7.5

To visualize:



95% Rule

From the fact that $\bar{X}_n \sim N(\mu, \frac{\sigma}{\sqrt{n}})$, we have:

$$P(-z_{\alpha/2} \leq \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \leq z_{\alpha/2}) = 1 - \alpha,$$

where $-z_{\alpha/2}$ is the critical value with a left-tail probability of 2.5%. With some math, we see:

$$P(-z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \bar{X}_n - \mu \leq z_{\alpha/2}\frac{\sigma}{\sqrt{n}}) = 1 - \alpha$$

$$P(\mu - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \bar{X}_n \leq \mu + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}) = 1 - \alpha$$

That is, $1 - \alpha$% of the time, the sample mean falls between $\mu - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$ and $\mu + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$.

Central Idea of Confidence Interval

The interval we defined above involves μ which we do not know! Instead, let's do the math slightly differently:

$$P(-z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \bar{X}_n - \mu \leq z_{\alpha/2}\frac{\sigma}{\sqrt{n}}) = 1 - \alpha$$

$$P(-\bar{X}_n - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq -\mu \leq -\bar{X}_n + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}) = 1 - \alpha$$

$$P(\bar{X}_n - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{X}_n + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}) = 1 - \alpha$$

Now we are able to categorize some uncertainty surrounding μ. In particular, we have that in repeated sampling of $\bar{X}_n$ (many samples of size n), μ will be in the interval

$$\left[\bar{X}_n - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{X}_n + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right]$$

$1 - \alpha$% of the time.

Confidence Interval

The 95 part of the 68-95-99.7 rule for Normal distributions says that $\bar{X}_{654}$ is within 2 standard deviations of the mean μ in 95% of samples. Therefore $z_{\alpha/2} \approx 2$.

Since $\sigma = 7.5$, the standard deviation of our sampling distribution is $\frac{7.5}{\sqrt{654}} = 0.3$ (from the Central Limit Theorem).

This means for **95% of all samples of size 654**, the sample mean $\bar{X}_{654}$ is within 0.6 of the population mean (two std deviations).

If we estimate that μ lies somewhere in the interval from $\bar{X}_{654} - 0.6$ to $\bar{X}_{654} + 0.6$, we'll be right for 95% of all possible samples.

Confidence Interval

Plugging in all the information we have leads to:

$$\left[26.8 - 0.6, 26.8 + 0.6\right] = \left[26.2, 27.4\right]$$

This interval is a **confidence interval**, which says that we are *95% confident* that the true mean of the BMI, μ, is in between 26.2 and 27.4 *

* This is because we got this interval from a method that captures the population mean for 95% of all possible samples

Confidence Intervals

Let's see what I mean. Let's say the population average ACT Score is 27. I will draw many samples of size 654 from the distribution $N(27, 7.5)$.

For each sample, I will calculate $\bar{X}_{654}$ and add and subtract 0.6 to form my confidence interval $[\bar{X}_{654} - 0.6, \bar{X}_{654} + 0.6]$.

Confidence Intervals

In that example, the confidence interval was $\bar{X}_{654} \pm 0.6$. In general, a confidence interval takes the form

$$\text{estimate} \pm \text{margin of error}$$

where the margin of error shows how much variability there is in our estimate

Margins of Error

For a given level of confidence, C (say 95%), the margin of error for our sample mean as:

\daisy{k} = Z_{\frac{1-C}{2}} \frac{\sigma}{\sqrt{n}}

- Let's say $C = 95\%$. We want to capture the middle 95%, so $\frac{1-.95}{2} = 2.5\%$ in each tail.
- $Z_{0.025} = 1.96$ which is where the ~ 2 standard deviations comes from.
- 90% Confidence Interval: $\implies Z_{\frac{1-C}{2}} = Z_{.05} = 1.645$ standard deviations.

Example

Lets determine the *exact* margin of error for previous example

$$k = Z_{\frac{1-C}{2}} \cdot \frac{\sigma}{\sqrt{n}}$$

If we are calculating 95% confidence interval, where $\bar{X}_{654} = 26.8$ $\sigma = 7.5$, then

$$k = Z_{0.025} \cdot \frac{7.5}{\sqrt{654}}$$

We find $Z_{0.025}$ using the table. $Z_{0.025}$ is the z-score such that $P(z > Z_{0.025}) = 0.025$

- Look up 0.025 (or 0.975!) and find the corresponding z-score

Example

Using the z-table, we find that $Z_{0.025} = 1.96$. This means:

$$k = 1.96 \cdot \frac{7.5}{\sqrt{654}} = 0.57$$

This means our *exact* 95% confidence interval is:

$$[26.23, 27.37]$$

Clicker Question

What Z-score will be associated with a 82% confidence interval

a. 0.92
b. 1.34
c. 0.82
d. 0.79

Example

Say high-school freshmen are sampled to see how long they spend on social media per day. The sample mean of $n = 50$ students is $\bar{X}_{50} = 2.1$ hours. The standard deviation of the population is 0.5 hours.

What is the 90% confidence interval for our estimate of the mean number of hours per day?

Clicker Question

A 95% confidence interval for the mean hours freshmen spent on social media per day was calculated to be [2.5 hours, 3.1 hours] based off a sample mean $\bar{X}_{50} = 2.8$. The confidence interval was based on a SRS sample of $n = 50$.

The standard deviation of the population is:

a. 0.3
b. 1.96
c. 0.2772
d. 1.0823

Margins of Error

So to recap, the margin of error is calculated by:

$$k = Z_{\frac{1-C}{2}} \cdot \frac{\sigma}{\sqrt{n}}$$

This means the size of the margin of error is determined by:

- level of confidence
- size of sample (can sometimes control)
- variance (which we can't control)

Discuss with a partner what happens to margin of error when

- increase the level of confidence required
- decrease sample size
- increase variance of population

Margins of Error

Level of confidence

- higher the level of confidence we want to have, larger the margin of error

Size of sample

- larger the sample, smaller the margin of error

- Variance

larger the variance, larger the margin of error

Example

Given a sample mean of 8, from a sample of 36 observations, and a variance of 25, construct 90%, 95%, and 99% confidence intervals for the true value, μ.

Confidence Intervals

How to **properly** think of a confidence interval:

1. You collect many different samples from the population
2. For each sample, you calculate the mean and the associated confidence interval around that mean
3. You'll have a confidence interval for each sample you collected
4. **95% of the confidence intervals you calculated will include the true mean `(\mu)`**

Confidence Intervals

Common misconceptions about confidence intervals

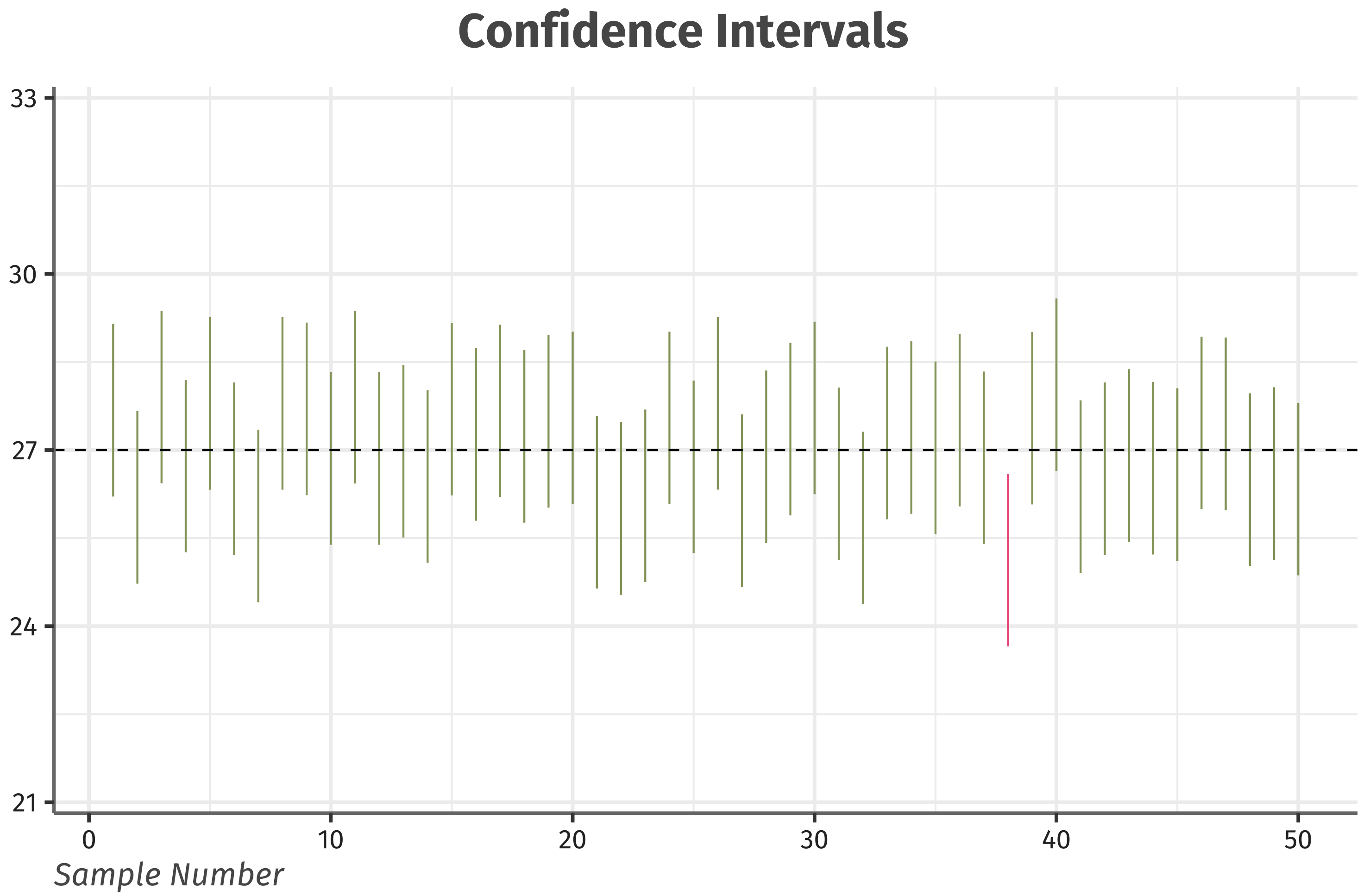
The CI **does not** tell you:

- The true mean is inside the confidence interval
- The probability the true mean is in your CI is C%

The CI **does** tell you:

- the range of estimates that contain the true mean C% of the time *in repeated sampling*

Confidence Intervals



Clicker Question -- Midterm Example

Veterinary researchers at a major university veterinary hospital calculated a 99% confidence interval for the average age of horses admitted for laminitis (a particular foot disease) as 6.3 to 7.4 years.

Based on this information we conclude that:

a. 99% of all horses admitted for laminitis are between 6.3 and 7.4 years old
b. 99% of the time, the average age of a horse admitted for laminitis will be between 6.3 and 7.4 years
c. We are 99% confident that the true mean age of horses with laminitis is between 6.3 and 7.4 years old
d. 99% of all samples of size n=25 will have an average age of horses with laminitis between 6.3 and 7.4 years old.