Euler graphs, triangle-free graphs and bipartite graphs in switching classes

Jurriaan Hage
Center for Software Technology, Utrecht University
jur@cs.uu.nl, http://www.cs.uu.nl/people/jur

Joint work with
Tero Harju (Dept. of Mathematics, University of Turku)
Emo Welzl (Institut für Theoretische Informatik, ETH Zentrum)

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If you need motivation...

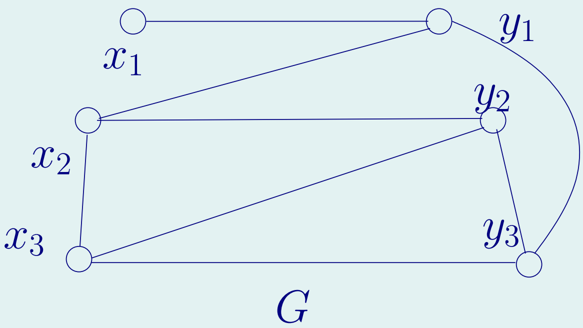
- The operation of switching has propped up in various places:
  - sociology: modelling relations between people in a group
  - mathematics: equidistant point sets in elliptic geometry and many more
  - computer science: formalizing certain types of networks of processors
- Global transformations of a graph are achieved by local transformations
- The local transformation are, in general, group actions.
- Today only the case where the group is addition modulo 2,
- and only computational/algorithmic aspects
- Because they are there.

Graphs

- Graph $G = (V, E)$
- Set of vertices $V = V(G)$
- Set of nonreflexive, undirected edges $E = E(G)$.
- Vertices have an identity.
- Restriction of G to vertex set $X \subseteq V$: $G|_X$.

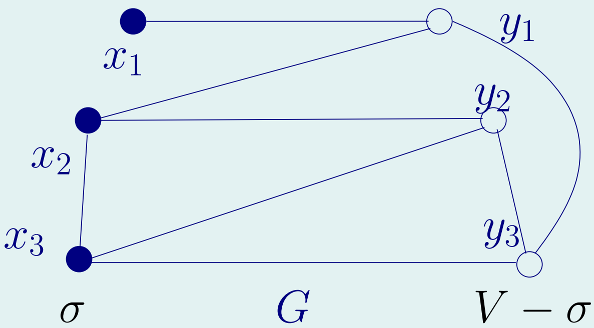
Switching classes definition

- **Take a graph G**



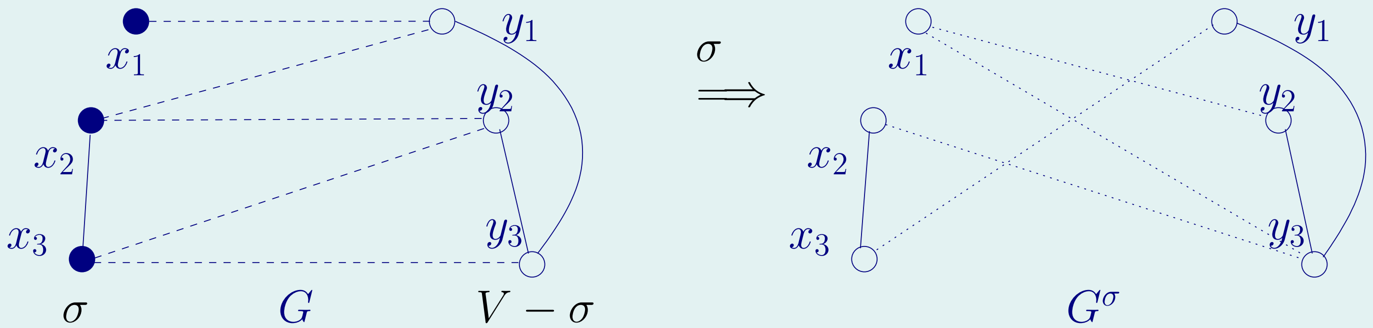
Switching classes definition

- Take a graph G, **select a set of vertices, a selector,** σ



Switching classes definition

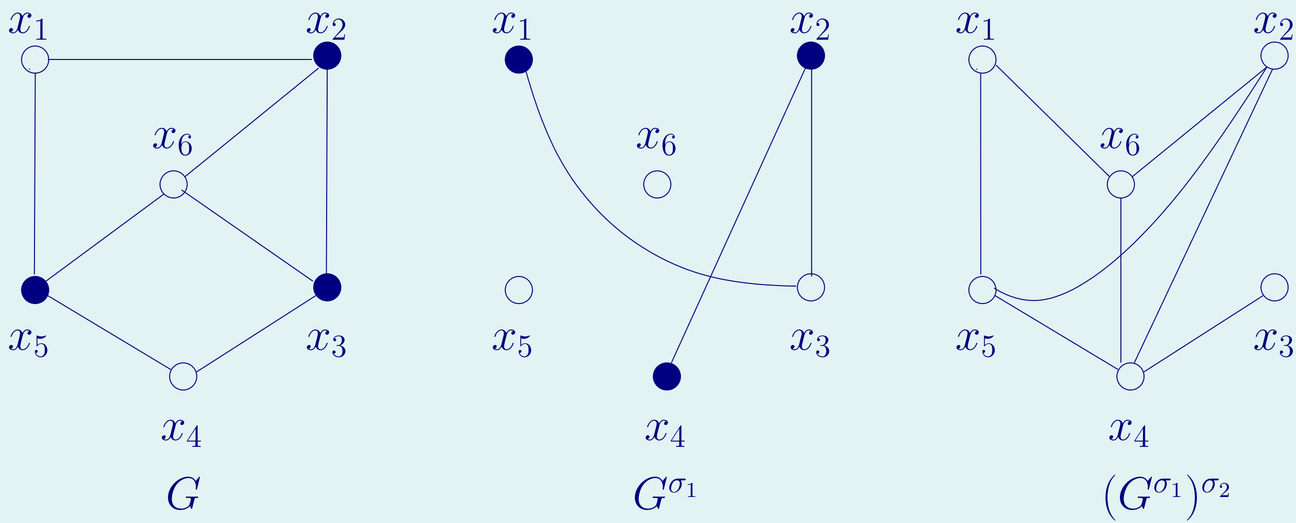
- Take a graph G, select a set of vertices, a selector, σ, **perform the switch**



- $G|_\sigma$ and $G|_{V-\sigma}$ do not change.
- Closed under complement, $(G^\sigma)^\sigma = G$.
- Switching class generated by G: $[G] = \{G^\sigma \mid \sigma \subseteq V(G)\}$.

A switching class is an equivalence class

- $G^{\emptyset} = G$ (reflexivity) and $(G^{\sigma})^{\sigma} = G$ (symmetry).
- Transitivity: $(G^{\sigma_1})^{\sigma_2} = G^{\sigma}$ for $\sigma = (\sigma_1 - \sigma_2) \cup (\sigma_2 - \sigma_1)$.
  Example: $\sigma_1 = \{x_2, x_3, x_5\}$ and $\sigma_2 = \{x_1, x_2, x_4\}$.
  Symmetric difference yields selector $\{x_1, x_3, x_4, x_5\}$.



Some useful results

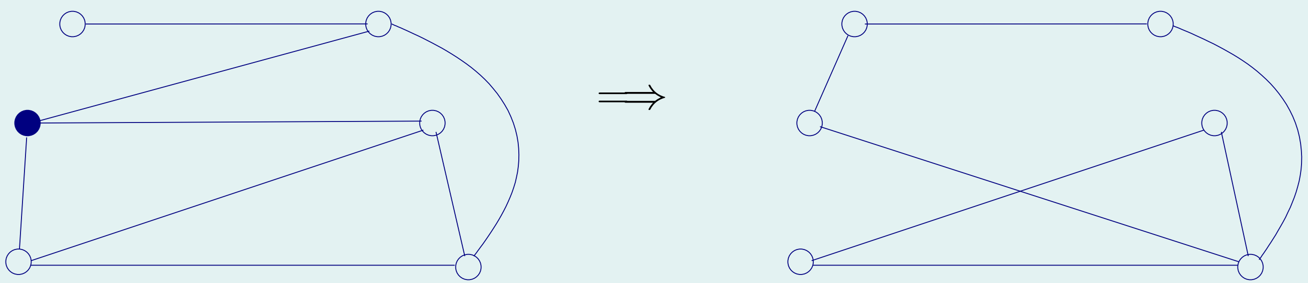
- Lemma: Adding G and G^σ edgewise (modulo 2/xor) yields the complete bipartite graph $K_{\sigma,V(G)-\sigma} = G + G^\sigma$.
- Proof: Switching complements edges between vertices from σ and $V - \sigma$, so each of these edges is either in G or G^σ, but never both.
  All other edges are present in both: they drop out.[Example]
- The main tool for people working in switching is the following:
  - Theorem: For every vertex $u \in V(G)$ and $A \subseteq V(G) - u$, there is a unique graph $H \in [G]$ where u has exactly the vertices in A as neighbours.
  - Proof existence: $H = (G^{N(u)})^A$ [Example]
  - Proof uniqueness: compute $H' + H$, where H' is another one.
    * $H' + H$ is a complete bipartite graph
    * u has no neighbours in $H' + H$

Three problems

- Instead of "Is G a graph of a given type?",
  we ask "Does $[G]$ contain a graph of a given type?"
- A switching class contains an exponential number of graphs.
- In the paper we considered three different types
  - eulerian
  - triangle-free
  - (bipartite)
- What is the complexity of these three problems for switching classes?
- They all turn out to be polynomial in the number of vertices of G.
- Earlier results we had do not help.
- The resulting algorithms can help during further research, e.g.,
  - characterizing the switching classes which contain a graph of a certain type by a list of forbidden subgraphs.

Switching classes with even graphs

- Eulerian: path that traverses each edge once and returns to the starting vertex.
- Equivalent to having only vertices of even degree and being connected.
- Generalize to *even* graphs: graphs with components that are eulerian.
- Seidel proved: for graphs of *odd order* (that is, with an odd number of vertices), every switching class contains a unique even graph.
- For graphs of *even order*, either none are even, or half of them are even, the other half are *odd*.



- odd $\Longrightarrow$ even

Switching classes with eulerian graphs

- For $G = (V, E)$, does $[G]$ contain a connected even graph?
- Odd order: use Seidel's constructive proof and check for connectedness
- Even order: if $[G]$ has even graphs, then $[G]$ contains an eulerian graph unless the complete graph $K_V \in [G]$.
- K_V is the only connected graph in $[K_V]$ and it is odd, not even.
- Proof:
  - Let G be even, assume v to be isolated.
    - $*$ There is always an even G with such a v.
  - Find x, y in $G - v$ such that x and y **not** adjacent.
  - Now, $G^{\{x,v\}}$ is even and connected.
  - Does not work if x and y do not exist, but then $K_V \in [G]$.

Triangle-free switching classes

- G is triangle-free iff it does not contain any induced triangle K_3.
- K_5 has switches $K_4 \cup K_1$ and $K_3 \cup K_2$: no triangle-free switches
- Polynomially reduce the decision problem

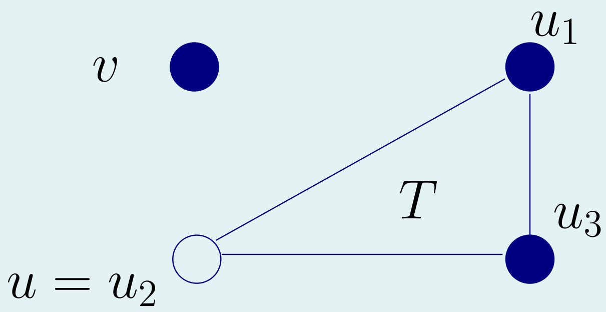
  "Does [G] contain a triangle-free graph?"

  to 2SAT (which has a polynomial solution).
- 2SAT: every formula is a conjunction of disjunctions where every disjunction contains at most two literals: $\phi = \overline{u} \wedge (u \vee v)$
- Selector σ satisfies 2SAT formula $\phi \iff \sigma(u) = 0$ and $\sigma(v) = 1$.

The basic situation

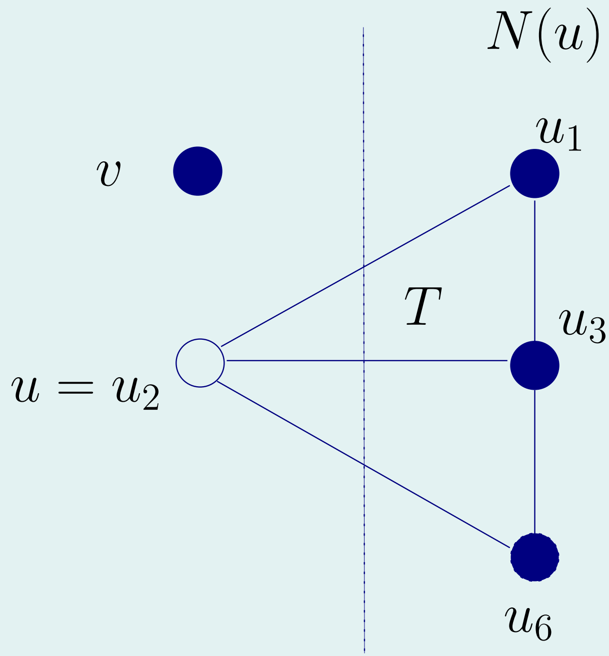
- Assume G contains an isolated vertex v and set $\sigma(v) = 1$ (no restriction).
- If G contains K_4, then answer **no**.
- If G is triangle-free then answer **yes**.
- Construct a formula, satisfiable by a selector σ only if G^σ is triangle-free. It guarantees all triangles are removed, and no new ones introduced.
- We need to consider only one arbitrary triangle.
- So, let $T = \{u_1, u_2, u_3\}$ be any triangle.
- To remove T, exactly one or two vertices of T must be switched.
- However, switching only one of them gives a triangle with v and the two others.

Constructing the formula: the triangle



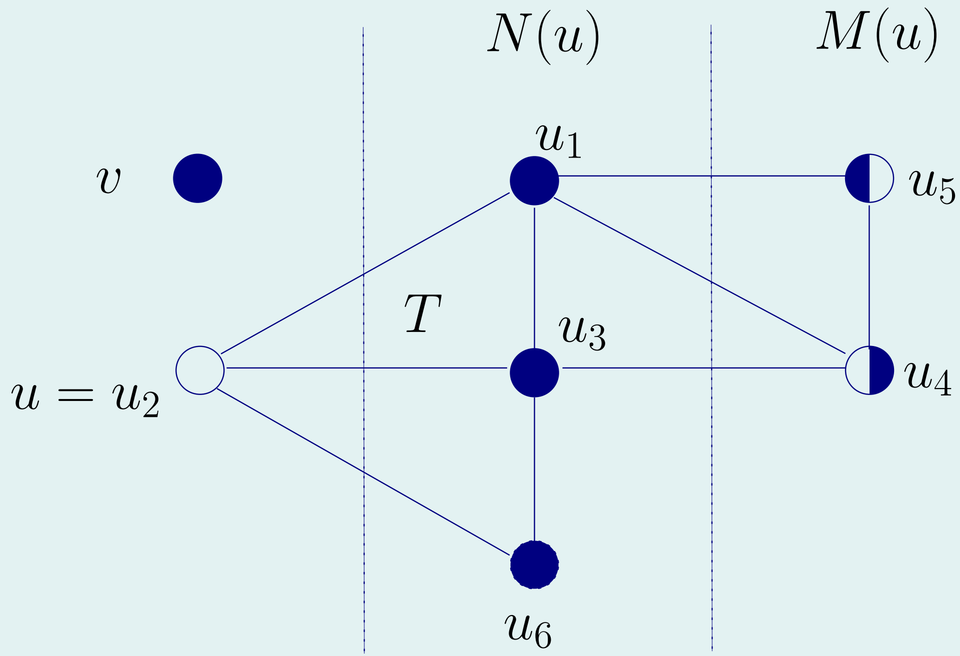
- Formula so far: $\mathbf{v}\,\overline{\mathbf{u_2}}\,\mathbf{u_1}\,\mathbf{u_3}$
- In this example we chose $u = u_2$.

Constructing the formula: all neighbours of u_2



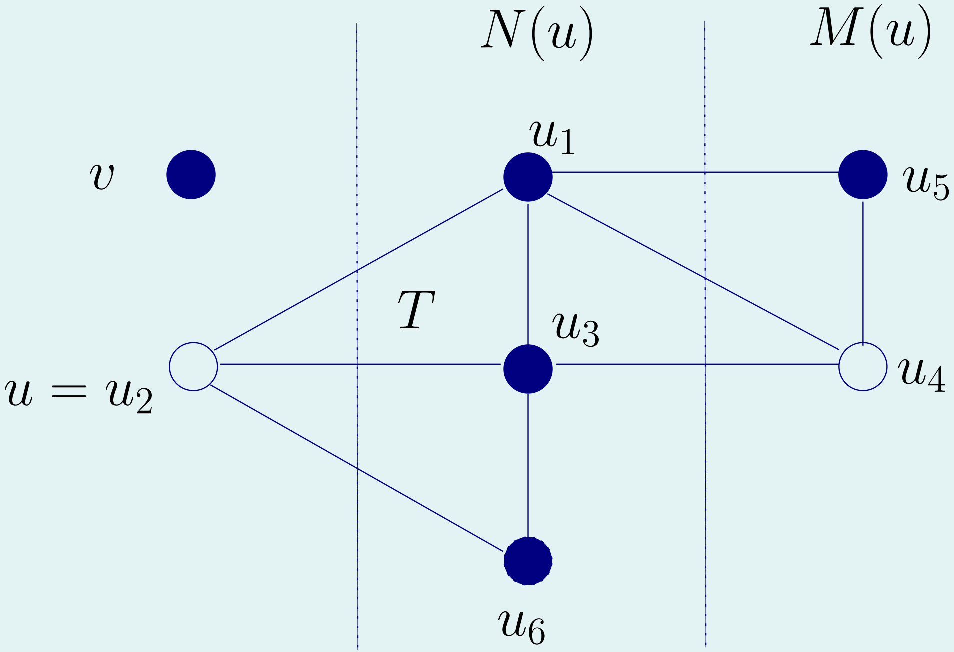
- Formula so far: $v\,\overline{u_2}\,u_1\,u_3\,\mathbf{u_6}$

Constructing the formula: a bipartite subgraph



- Formula so far: $v\,\overline{u_2}\,u_1\,u_3\,u_6\,(\mathbf{u_4} \vee \mathbf{u_5})(\overline{\mathbf{u_4}} \vee \overline{\mathbf{u_5}})$

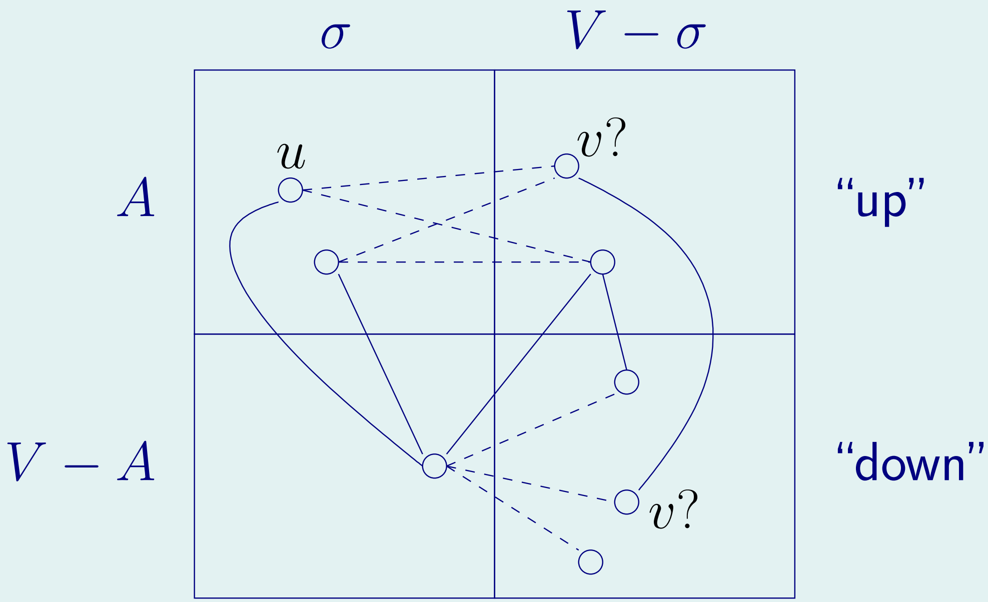
Constructing the formula: the final triangles



- Final formula: $v\,\overline{u_2}\,u_1\,u_3\,u_6\,(u_4 \vee u_5)(\overline{u_4} \vee \overline{u_5})\,\overline{\mathbf{u_4}}\,\mathbf{u_5}$
- $\overline{u_4}$ because of $\{u_1, u_3, u_4\}$
- u_5 because of $\{u_3, u_5, u_6\}$

Bipartite switching classes

- Solved similarly by reduction to 2SAT
- Main idea: every bipartite graph switches to a 4-colourable graph such that it consists of two disjoint, arbitrarily connected complete bipartite subgraphs.



Our approach

- Fix an arbitrary vertex u to go "up" (no restriction).
- 2SAT formula specifies which vertex must go down if u is up.
  - Note: in the triangle-free case it specified a selector.
- Same for a vertex v which goes down (this is a restriction).
- Hence, we must try all possible choices for v.
- If none works, then G is a not a switch of a bipartite graph.
- Hence we solve a linear number of 2SAT problems.
- We can do a little better: consider $\{u, v_1, v_2\}$ which induces a subgraph with an odd number of edges.
  - if u goes up then one of the others must go go down.
- Only two 2SAT problems need to be solved.

Conclusion and future work?

- Three non-trivial algorithms for problems lifted from graphs to switching classes. All are polynomial.
- Hard to find algorithms that become NP-complete.
  - In fact, Kratochvíl and Nešetřil and Zýka are the only ones to do so.
- Algorithms can be used to solve a problem we could not yet solve:

  Characterize switching classes which do not have a bipartite graph by means of set of forbidden subgraphs.

- A characterization for triangle-free would be nice, but probably not by forbidden subgraphs.