An Evaluation of Central Bank Long-Term Non-Issuance

Joe Anderson

May 2024

Abstract

As part of its institutional toolkit, the United States Federal Reserve conducts open market operations by exchanging shot-term Fed-issued liabilities for non-Fed-issued assets. Importantly, it is legally disallowed from issuing any new liability with a maturity greater than 90 days.

A debt-manager concerned with being 'regular and predicable' and 'targeting debt costs' may issue a structure of government debt different from that which maximizes welfare. A welfare-maximizing central bank with unconstrained issuance always perfectly corrects this structure, though one unable to issue long-term debt may not. I investigate theoretical consequences of market incompleteness resulting from this institutional constraint and find substantial welfare costs using a neural network solution method.

1 Introduction

Countries that separate their central bank from the larger legislature enjoy insulated monetary policy shielded from the explicit political pressures influencing fiscal decisionmaking.

This isolation is associated with better outcomes in inflation.[1] It also comes with a division of institutional objectives.[2] For instance, a country's debt-manager may be responsible for financing federal expenditures and minimizing borrowing costs while the central bank may follow a mandate of maximum employment and price stability, as is the case in the US.

Though isolated, central banks and their fiscal counterparts are tightly linked together through three channels. First, the central government's budget includes central bank remittances as an income source - central banks send all operating profits directly to the sovereign debt manager. The government uses these resources to relieve pressure on the taxpayer. Second, both institutions heavily use markets for federal debt to achieve their objectives. Governments issue and redeem their own debt, while central banks engage in open-market operations (exchanging government liabilities of one maturity for government liabilities of another). Not only do debt price movements affect both institutions, both authorities can influence these movements anywhere on the yield curve. A third and important point, though one that is not explored in this paper, is that "central bank isolation" as a statement can be viewed only as an approximation. Laws, mandates and charters can be changed by future legislatures, making true isolation a fantasy in our dynamic, strategic world.

This paper takes a close look at interactions between an optimizing debt-manager and an optimizing central bank that uses open-market operations. Both institutions have independent objectives resulting in an infinitely-repeated, non-cooperative game. Strategic interactions occur in the market for public debt along every maturity and a subgame perfect Nash equilibrium (SPNE) emerges.

As in the model in Lucas and Stokey (1983), there is a distortionary income tax rate which the government uses (along with real, non-contingent debt) to finance stochastic spending. A finite number of maturities exist for government debt which are imperfect substitutes with one another. Debt prices do not perfectly co-move upon realizing a new state. As a result, the representative household optimally constructs a portfolio of debt to manage its risk - debt's maturity structure

[1] As shown in Alesina and Summers (1993)

[2] As discussed in Stella and Lonnberg (2008).

takes on an insurance role.

A Ramsey-planner government commits to a tax and debt-supply plan such that household welfare is maximized once goods, labor and asset markets clear. Allocations match those from a complete markets economy if the maturity structure is sufficiently rich.[3] A government comprised of two simultaneous-moving institutions implements the Ramsey planner's solution as long as at least one institution (1) maximizes household welfare and (2) faces no constraints other than its budget constraint. A government with two such institutions faces a continuum of possible implementations of the Ramsey solution.

Finally, the model's welfare-maximizing central bank is precluded from issuing its own long-term debt in an environment where the model's debt-manager is targeting interest payments on issued debt. The calibrated model's constrained equilibrium is computed and contrasted with the unconstrained, coordinated government's solution. These equilibria may be far away from each other depending on the debt-manager's targets. The constrained economy loses 55% of the welfare difference between the Ramsey solution and total financial autarky in the calibrated model. This result reinvigorates discussion from Fisher (1932) that supports the allowance of central bank-issued long-term debt in the US.

The analysis builds on two strands of work. First, frameworks from the large literature on optimal management of non-contingent public debt lend themselves nicely to studying open-market operations as a policy tool. The complete markets framework found in Lucas and Stokey (1983), Angeletos (2002), Buera and Nicolini (2004) forms the basis of this paper's model. And incomplete markets papers such as Barro (1979), Aiyagari et al. (2002), Bhandari et al. (2017), when extended to include long-term debt, provide insight to results found later in this analysis.

Second, this research adds to the literature on non-cooperative games played by monetary and fiscal policymakers. Papers like Pindyck (1976), Blinder (1982), Nordhaus (1994) and Greenwood et al. (2014) lay the coordination problem out bare but analyze it without using DSGE models, which have become the standard in theoretical macroeconomic work. Other work by Kirsanova, Stehn, and Vines (2005) and Saulo, Rêgo, and Divino (2013) extend the New-Keynesian model to study non-cooperative strategic interactions when fiscal spending is a choice.

Exploring how the equilibrium maturity structure of publicly-held debt evolves with joint

[3]This is proven in Angeletos (2002) and explored further in Buera and Nicolini (2004).

movements in fiscal policy's issuance/redemption decisions and monetary policy's open-market operations has not previously been done within either the public debt or the non-cooperative policy games literature. The analysis in this paper differs from previous work along four main dimensions.

First, the game in this analysis is played between the debt-manager (e.g. US Treasury) and central bank (e.g. US Federal Reserve). Not between the legislature (e.g. Congress) and the central bank. This is an important distinction because treasuries, while fiscal agents, do not make spending decisions - they are constrained to take spending as given and optimize debt supply.[4]

The second main deviation from the existing work on policy non-coordination comes from the analysis's inclusion of time-varying, shape-varying debt maturity. It is difficult for a New Keynesian model to explore endogenous variation in debt structure because of the technical challenges when log-linearizing separate maturities in the household's budget constraint.

Third, unlike in previous non-cooperation research, the debt-manager's objective function is not just a re-weighting of the social loss function. Here, the debt-manager's objective matches stated directives from the US Treasury to "minimize debt servicing costs" and to be "regular and predictable."

Fourth and finally, this research directly explores how institution-specific asymmetries affect the game. For instance, how does the game change when the central bank is unable to issue long-term government debt (as is the case in the US) or when there is a chance that institutional remittance (recapitalization) payments don't occur (as is the case in many developing countries)? These are theoretical stones that have yet to be overturned.

2 Model

2.1 Model Environment

The model closely resembles Lucas and Stokey (1983)'s barter economy. There exists an infinitely-lived economy, where discrete periods are indexed by t. Three agent types inhabit the model: households, a debt-manager and a central bank.

A measure-1 continuum of identical households consume c_t and produce an aggregated good in

[4]In this paper, decisions made by the legislature follow a stochastic process.

every period equal to their labor supply n_t. They may also lend or borrow using a portfolio of indexed (real-valued), non-contingent government bonds $b_t = \left\{ b_t^{(t+j)} \right\}_j$, where $j \in \{j_1 = 1, j_2, \cdots, j_i, \cdots, j_{I-1}, j_I = J\}$ represents a bond's term to maturity for $i \in \{1, \cdots, I\}$. I represents the number of debt instruments issued each period and J represents the maximum maturity of the portfolio. $I = J$ when each maturity is separated from the previous maturity by one time period, or $j_i - j_{i-1} = 1 \ \forall i > 1$. Otherwise, $J > I$.

Government spending is lump-sum, wasteful, and given as g_t every period. There exists a stochastic exogenous finite Markov process over spending $\{g_t\}_t$. Specify this process to be an S-state Markov chain, with transition matrix P. Call the vector of spending amounts $g \equiv \{g(s)\}_s$.

The government is split into two branches: the debt-manager and central bank. The debt-manager sets the per-period supply of each of the various debt assets $\hat{b}_t = \left\{ \hat{b}_t^{(t+j)} \right\}_j$ in a given period through new issuance and debt redemption. The central bank demands debt at all maturities given by $\tilde{b}_t = \left\{ \tilde{b}_t^{(t+j)} \right\}_j$ through debt purchases (and offsetting sales).[5]

A linear (distortionary) household labor tax rate τ_t is passively set to balance the government's consolidated budget constraint each period, given g_t, $\hat{b}_t$ and $\tilde{b}_t$.

2.2 Market Structure

I asset markets exist in every period: one for each debt instrument in existence. Debt is exchanged at prices $q_t = \left\{ q_t^{(t+j)} \right\}_j$ where $q_t^{(t+j)}$ is the price of a bond that matures in period $t+j$. Agents borrow (lend) $q_t^{(t+j)}$ units of the consumption good in period t and pay (receive) 1 unit of the good in period $t+j$, so that $q_{t+j}^{(t+j)} = 1 \ \ \forall\ t,\ j$.

2.3 Households

A (price-taking) household maximizes discounted expected utility over its infinite-period life:

$$\max_{c_t, n_t, b_t} \left\{ \mathbb{E} \sum_t^{\infty} \beta^t u(c_t, n_t) \right\}$$

[5] Some might be more comfortable thinking of central bank debt decisions as additional issuance or redemption. This would reverse the signs of central bank allocations here. In this paper, sign of the central bank's debt decisions are consistent with its balance sheet positions. For example, positive debt in the model reflects higher central bank assets, although it implies less outstanding supply held by households.

$$s.t. \quad c_t + \sum_{j=1}^{J} q_t^{(t+j)} \left(b_t^{(t+j)} - b_{t-1}^{(t+j)} \right) = (1 - \tau_t) \, n_t + b_{t-1}^{(t)} \tag{1}$$

$$s.t. \quad \sum_{j=1}^{J} q_t^{(t+j)} \left(b_t^{(t+j)} - b_{t-1}^{(t+j)} \right) \geq -L \tag{2}$$

where L is an arbitrarily large number.

The LHS of the household's budget constraint represents the expenditures of the household (spending on the consumption good and the sum of household lending across all debt markets). The RHS represents household inflows in the period (after-tax labor income and income from matured debt).

The second constraint is a borrowing constraint on the household, included to eliminate household ponzi strategies. L is set such that this constraint does not bind in equilibrium.

2.4 Government

A consolidated government that aims to maximize household utility achieves the economy's efficient benchmark. Replacing the consolidated government with a debt-manager and central bank that individually and non-cooperatively achieve independent objectives yields additional insight.

2.4.1 Government 1 - Benevolent Consolidated Government

Define the benevolent consolidated government as a Ramsey planner constrained by the government's consolidated budget constraint. Define the net debt allocation for maturity j as $\boldsymbol{b}_t^{(t+j)} = \hat{b}_t^{(t+j)} - \tilde{b}_t^{(t+j)}$ and define $\boldsymbol{b}_t = \left\{ \boldsymbol{b}_t^{(t+j)} \right\}_j$.[6] Write the benevolent consolidated government's problem as:

$$\max_{\boldsymbol{b}_t} \left\{ \mathbb{E} \sum_{t}^{\infty} \beta^t u \left(c_t, n_t \right) \right\}$$

$$s.t. \quad \boldsymbol{b}_{t-1}^{(t)} + g_t = \tau_t n_t + \sum_{j=1}^{J} q_t^{(t+j)} \left(\boldsymbol{b}_t^{(t+j)} - \boldsymbol{b}_{t-1}^{(t+j)} \right) \tag{3}$$

where the government takes optimal household and firm behavior as additional constraints.

[6] $\boldsymbol{b}_t$ is a difference in institutional allocations because the debt-manager supplies debt while the central bank demands it. The net of the two is what the government jointly supplies.

2.4.2 Government 2 - Two Institutions

Debt-Manager. The debt-manager's problem is given as:

$$\max_{\hat{b}_t} \{V_{dm}\}$$

$$s.t.\;\; \hat{b}_{t-1}^{(t)} + g_t = \tau_t n_t + \sum_{j=1}^{J} q_t^{(t+j)} \left(\hat{b}_t^{(t+j)} - \hat{b}_{t-1}^{(t+j)} \right) + \Delta_t \tag{4}$$

taking household and central bank optimization and market-clearing as additional constraints. Δ_t represents remittance payments from the central bank.

Central Bank. The central bank solves its problem:

$$\max_{\tilde{b}_t} \{V_{cb}\}$$

$$s.t.\;\; \tilde{b}_{t-1}^{(t)} = \sum_{j=1}^{J} q_t^{(t+j)} \left(\tilde{b}_t^{(t+j)} - \tilde{b}_{t-1}^{(t+j)} \right) + \Delta_t \tag{5}$$

while taking household and debt-manager optimization and market-clearing as additional constraints.

Strategies. This model abstracts from private and/or hidden information. When institutions move simultaneously, denote the debt-manager's feasible strategy space at a particular state given the central bank's choice of $\tilde{b}_t$ as $\Gamma_{dm}\left(s, \boldsymbol{b}_{t-1} | \tilde{b}_t\right)$, where an individual feasible strategy at time t is $\hat{b}_t \in \Gamma_{dm}$. Denote the central bank's feasible strategy space at a particular state given the debt-manager's choice of $\hat{b}_t$ as $\Gamma_{cb}\left(s, \boldsymbol{b}_{t-1} | \hat{b}_t\right)$, where an individual feasible strategy at time t is $\tilde{b}_t \in \Gamma_{cb}$.[7]

The result of the model is a repeated game of complete information between the debt-manager and central bank. Uncertainty stems from a known aggregate process. Rational expectations characterize the beliefs of all agents.

[7] When movement is not simultaneous, the first mover conditions its decision on the best response function of the second mover rather than its opponents debt portfolio choice.

Combining Constraints. Notice that combining the debt-manager's and central bank's budget constraints (4), (5) through Δ_t reveals the government's consolidated budget constraint (3):

$$\left(\hat{b}_{t-1}^{(t)} - \tilde{b}_{t-1}^{(t)}\right) + g_t = \tau_t n_t + \sum_{j=1}^{J} q_t^{(t+j)} \left(\left(\hat{b}_t^{(t+j)} - \tilde{b}_t^{(t+j)}\right) - \left(\hat{b}_{t-1}^{(t+j)} - \tilde{b}_{t-1}^{(t+j)}\right)\right) \tag{6}$$

2.5 Market-Clearing

The goods market clears when total production is consumed by households and government:

$$n_t = c_t + g_t \tag{7}$$

The debt market for maturity $j \in \{1, \cdots, J\}$ clears when the amount of debt supplied jointly by the debt-manager and central bank equals the amount of debt demanded by households:

$$\hat{b}_t^{(t+j)} - \tilde{b}_t^{(t+j)} = \boldsymbol{b}_t^{(t+j)} = b_t^{(t+j)} \quad \forall j \tag{8}$$

3 Definition: Competitive (Subgame Perfect Nash) Equilibrium

Given state variables $(\boldsymbol{b}_{t-1}, g_t)$, a stochastic process for spending (P) and well-defined government policy $(\boldsymbol{b}_t,\ \tau_t,\ \Delta_t)$ resulting from strategies $\left(\hat{b}_t,\ \tilde{b}_t\right)$ consistent with rational expectations and realizations of allocations, a competitive equilibrium is a set of prices (q_t) and quantities $(c_t,\ n_t,\ b_t)$ for all t such that:

1. Households, the debt-manager and the central bank all solve their constrained optimization problems.

2. The goods market and all J debt markets clear for all t, so that (7) and (8) hold.

3. The consolidated government budget constraint is satisfied, so that (3) holds.

4 The Model Under Effectively Complete Markets

Begin by analyzing the model assuming a maturity structure whereby allocations consistent with effectively complete markets are feasible given welfare-maximizing management of such a structure. The model is solved according to the timeless perspective proposed in Woodford (1999) here.

4.1 Institutional Equivalence

Lucas and Stokey (1983) prove that a maturity structure of non-contingent debt exists that supports the planner's complete-market solution in a time-consistent manner, so long as the structure is sufficiently rich.

Angeletos (2002) further refines this result, proving that each maturity $\{1, \cdots, J\}$ is non-redundant and that a unique and time-invariant welfare-maximizing maturity structure exists if the number of debt maturities I is equal to the number of possible exogenous states S. As in Lucas and Stokey (1983), the allocations resulting from the welfare-maximizing structure where $I = S$ match those in an economy with complete markets.

Impose $I = S$ for the rest of the paper. Further results are proven when the model is expanded to include a decentralized government. All proofs can be found in appendix A.

Lemma 1. *Given information of an arbitrary central bank strategy, the debt-manager can itself achieve the welfare-maximizing debt allocation subject to the consolidated GBC.*

Lemma 2. *Given information of an arbitrary debt-manager strategy, the central bank can itself achieve the welfare-maximizing debt allocation subject to the consolidated GBC.*

The intuition behind Lemmas 1 and 2 is that either institution's actions can be exactly 'unwound' by the other. So if the central bank is, for whatever reason, exogenously mandated to purchase a certain portfolio of debt, the debt-manager can perfectly offset this purchase at every available maturity by additional issuance and/or redemption so that the welfare-maximizing debt supply is achieved (and vice versa).

Proposition 1. *Net debt allocations resulting from a simultaneous-move government with at least one welfare-maximizing institution are equivalent to net debt allocations resulting from choices of* *a consolidated government with a welfare-maximizing objective function and constrained by the consolidated GBC, so long as a SPNE exists.*

The proposition fully equates the planner's problem to the decentralized government's problem in the supply of government debt to the household (the difference of issued and central-bank-demanded debt) when even just one institution maximizes social welfare.

A subgame perfect Nash equilibrium (SPNE) is not guaranteed without restrictions on the non-welfare-maximizing institution's preferences. Uniqueness in the non-welfare-maximizing institution's choice guarantees uniqueness in its counterpart's action.

Corollary 1. *Any allocation resulting from Lemma 1 is an equilibrium for a welfare-maximizing debt-manager and welfare-maximizing central bank.*

Corollary 2. *Any allocation resulting from Lemma 2 is an equilibrium for a welfare-maximizing debt-manager and welfare-maximizing central bank.*

These corollaries speak to the SPNE in the game between two welfare-maximizing institutions. If one institution's behavior is held fixed such that the other perfectly achieves the welfare-maximizing maturity structure, then the first institution achieves its objective by keeping its behavior constant - even after it is free to adjust - constituting an equilibrium.

Lemma 3. *$I \times 2$ continua of within-governmental implementations of the efficient solution exist with welfare-maximizing objective functions across institutions.*

When all institutions maximize welfare, the same outstanding debt structure is targeted by all. While there is a unique equilibrium in $\boldsymbol{b}_t$, there an infinity of equilibria in each maturity of $\tilde{b}_t$ and $\hat{b}_t$ as both welfare-maximizing institutions are indifferent among specific intragovernmental ownership of assets.

4.2 The Complete Market Ramsey Plan

Proposition 1 states that the social planner's solution and a that of the problem with a welfare-maximizing institution are equivalent. As such, solve the centralized benevolent government's problem below with the intention to use it in the decentralized government's equilibrium.

For the remainder of the paper, model preferences as CRRA over consumption and labor:

$$u(c_t, n_t) = \frac{c_t^{1-\sigma}}{1-\sigma} - \frac{n_t^{1+\varphi}}{1+\varphi} \quad \text{where} \;\; \sigma \geq 0, \;\; \varphi \geq 0$$

where σ and φ are inverses of the intertemporal elasticity of substitution and Frisch elasticity of labor supply, respectively.[8]

Markets are complete from the assumptions on the number of maturities and states in the economy ($I = S$). The Ramsey planner solves for optimal allocations according to the problem:

$$\max_{\{c_t, n_t\}} \left\{ \mathbb{E} \sum_t^{\infty} \beta^t \left(\frac{c_t^{1-\sigma}}{1-\sigma} - \frac{n_t^{1+\varphi}}{1+\varphi} \right) \right\} \tag{9}$$

$$s.t. \;\; \mathbb{E} \sum_t^{\infty} \Lambda_t \left[(1-\tau_t)\, n_t - c_t \right] = 0 \tag{10}$$

where Λ_t is the household's stochastic discount factor. The implicit assumption on the constraint (10) is that net transfers (RHS) are zero coming into time t. This is done for computational convenience with no loss of generality.

Using Walras' Law with the aggregate resource constraint (7) and (10), the time-0 government budget constraint is derived as:

$$0 = \mathbb{E} \sum_t^{\infty} \Lambda_t \left(\tau_t n_t - g_t \right) \tag{11}$$

This equation can be interpreted two ways: that the expected present value of current and future government primary surpluses either sum to outstanding time t government transfers, or sum to the market value of the time $t-1$ debt portfolio entering period t (the two interpretations are interchangeable). Call (11) the government's solvency condition.

The Ramsey problem charges the planner to maximize welfare, (9), given the aggregate resource constraint (7) and the government's solvency constraint (11). A feasible allocation is a stochastic process $\{c_t, n_t\}$ that satisfies all three. The allocation that solves the Ramsey problem is called the Ramsey allocation. The tax plan that solves the Ramsey problem is called the Ramsey tax plan.

[8] While these preferences lead to counterfactual asset-pricing movements (and debt portfolio selection as a result), they keep the analytics tractable. A popular alternative to this specification is Epstein-Zin preferences.

Ramsey Roadmap. The problem is solved by executing the following steps, following Sargent and Velde (1999). This roadmap is worked through in appendix B.

1. Solve for the household FOCs and express the tax system and price system in terms of allocations.
2. Use these FOCs to express the government budget constraint as a constraint on allocations.
3. Find the FOCs of the Ramsey planner to characterize the feasible allocation that maximizes welfare subject to the restriction from step 2.
4. Use the household FOCs from step 1 along with the equilibrium allocations from step 3 to solve for the equilibrium Ramsey plan.

The tax plan that implements the Ramsey allocation is entirely forward looking - it consists of tax rates that vary only with the exogenous state variable $g(s)$. Express the Ramsey tax plan as $\tau^* = \{\tau(s)^*\}_s$ where $\tau_t^* \in \tau^* \ \forall t$.

4.3 Asset Pricing

Consumption is a function of current and expected future prices of debt from the household's optimal borrowing solution. In an economy where all debt is traded, the following equations hold:

$$c_t = \left(\frac{q_t^{(t+1)}}{\beta \mathbb{E}_t \left[c_{t+1}^{-\sigma}\right]}\right)^{\frac{1}{\sigma}} = \left(\frac{q_t^{(t+j)}}{\beta \mathbb{E}_t \left[c_{t+j}^{-\sigma}\right]}\right)^{\frac{1}{\sigma}} \quad \forall j \tag{12}$$

where each asset pays out one consumption good upon maturity with certainty $q_{t+j}^{(t+j)} = 1 \ \ \forall j$.

From (12) arises a no-arbitrage condition for assets in the economy. When all I debt maturities are traded, the price of a j-period asset equals the expected product of 1-period bond prices from period t to $t+j$:

$$q_t^{(t+j)} = q_t^{(t+1)} \mathbb{E}_t \left[q_{t+1}^{(t+j)}\right] = q_t^{(t+1)} \mathbb{E}_t \left[q_{t+1}^{(t+2)} q_{t+2}^{(t+3)} ... q_{t+j-1}^{(t+j)}\right] \tag{13}$$

Debt prices adjust in general equilibrium so that this condition holds. Combining the ARC (7), the HH FOC on labor, and the no-arbitrage condition (13) results in the system:

$$\begin{cases} q_t^{(t+j)} = \beta^j \dfrac{\mathbb{E}_t \left[c_{t+j}^{-\sigma} \right]}{c_t^{-\sigma}} \quad \forall j \\ c_t^{-\sigma} \left(1 - \tau_t\right) = \left(g_t + c_t\right)^{\varphi} \end{cases} \tag{14}$$

Debt prices cannot be solved closed-form analytically, though they are easily computed numerically given the process for $g(s)$ and the Ramsey tax plan $\{\tau(s)^*\}_s$.

4.4 Ramsey Plan's Supporting Debt Position

In order for the economy to proceed according to the Ramsey plan, debt of all maturities needs to be carefully issued by the government. Write the time-t equilibrium implementability constraint (implied by the equilibrium forward-iterated HH budget constraint) as:

$$z_t^* = \frac{\mathbb{E}_t \sum_{i=0}^{\infty} \beta^i \left[\left(n_{t+i}^* - g_{t+i}\right) c_{t+i}^{*-\sigma} - n_{t+i}^{*1+\varphi} \right]}{c_t^{*-\sigma}} \tag{15}$$

where $c_t^* = c(\tau(s)^*)$ and $n_t^* = n(\tau(s)^*)$ reference the Ramsey planner's choice allocations given the exogenous state $s \in \{1, \cdots, S\}$ being realized at time t.

$z_t^* = z(s)^*$ can again be thought of as either the time t market value of state-contingent debt issued by the government at time $t-1$ or the total amount of transfer payments the government makes to the public at time t along the Ramsey tax plan. It's important to note that there are no lump-sum transfers or taxes in this model, so z_t^* can only be achieved by the government through manipulation of the maturity structure.[9]

Lucas and Stokey (1983) show that a non-committing central government can construct a debt issuance plan that supports the plan τ^* and induce its successor to do the same. Buera and Nicolini (2004) abstracts from the time inconsistency problem but develops a method for backing out the time-invariant, welfare-maximizing maturity structure in the economy. This paper follows Buera and Nicolini (2004)'s lead for all complete-market specifications. The key to optimal management: the central government issues (redeems) a specific debt portfolio so that price changes in non-matured

[9]If lump sum taxes were added, a result similar to the one found in Aiyagari et al. (2002) would arise - a slow accumulation of assets (negative debt) over time until distortionary taxation is no longer required. Interest payments from the government asset position eventually entirely finance both government spending and Ramsey transfers.

debt brings the market price of the entire portfolio at time t in line with z_t^*.

Given $c(\tau(s)^*)$, $n(\tau(s)^*)$ and the stochastic process for g, the welfare-maximizing $z(s)^*$ is solved for in every state. Define $Z_t^* = Z^*\ \forall t$ as an $S \times 1$ vector of state-contingent wealth transfers from the government to households along the Ramsey plan:

$$Z_t^* = Z^* = \begin{bmatrix} z(1)^* & z(2)^* & \dots & z(S)^* \end{bmatrix}'$$

and define A_t^* as an $S \times J$ payout matrix, where each element is the ex-post market value of previously-issued j-period debt along the Ramsey plan:

$$A_t^* = \begin{bmatrix} 1 & \beta \dfrac{\mathbb{E}_t\left[c_{t+j_2-1}^{*^{-\sigma}} | s=1\right]}{c_t^{*^{-\sigma}}} & \dots & \beta^{J-1} \dfrac{\mathbb{E}_t\left[c_{t+J-1}^{*^{-\sigma}} | s=1\right]}{c_t^{*^{-\sigma}}} \\ 1 & \beta \dfrac{\mathbb{E}_t\left[c_{t+j_2-1}^{*^{-\sigma}} | s=2\right]}{c_t^{*^{-\sigma}}} & \dots & \beta^{J-1} \dfrac{\mathbb{E}_t\left[c_{t+J-1}^{*^{-\sigma}} | s=2\right]}{c_t^{*^{-\sigma}}} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \beta \dfrac{\mathbb{E}_t\left[c_{t+j_2-1}^{*^{-\sigma}} | s=S\right]}{c_t^{*^{-\sigma}}} & \dots & \beta^{J-1} \dfrac{\mathbb{E}_t\left[c_{t+J-1}^{*^{-\sigma}} | s=S\right]}{c_t^{*^{-\sigma}}} \end{bmatrix} = \begin{bmatrix} 1 & q_{t,1}^{(t+j_2-1)^*} & \dots & q_{t,1}^{(t+J-1)^*} \\ 1 & q_{t,2}^{(t+j_2-1)^*} & \dots & q_{t,2}^{(t+J-1)^*} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & q_{t,S}^{(t+j_2-1)^*} & \dots & q_{t,S}^{(t+J-1)^*} \end{bmatrix}$$

And because the nature of the stochastic process is first-order Markov, A_t^* is independent of history and, therefore, invariant with respect to the time period and the state.[10] A_t^* can be written as $A_t^* = A^*$.

Given Z^* and A^*, the following identity holds for government transfers to the public in each period along the Ramsey tax plan:

$$Z^* = A^* \boldsymbol{b}_{t-1}(s)^* \tag{16}$$

Premultiplying both sides of this equation by $A^{*^{-1}}$ yields the optimal maturity structure of non-contingent debt:

$$\boldsymbol{b}_{t-1}(s)^* = A^{*^{-1}} Z^*$$

[10]This is shown in Buera and Nicolini (2004).

Because both ${A^*}^{-1}$ and Z^* are constant matrices, the welfare-maximizing debt structure is both state-invariant and thus time-invariant:

$$\boldsymbol{b}_{t-1}(s)^* = \begin{bmatrix} \boldsymbol{b}_{t-1}^{(t)}(s)^* & \boldsymbol{b}_{t-1}^{(t+1)}(s)^* & \dots & \boldsymbol{b}_{t-1}^{(t+J-1)}(s)^* \end{bmatrix}' = \boldsymbol{b}^* \quad \forall t, s$$

4.5 Supporting Debt Positions for Feasible Tax Plans

Angeletos (2002) and Buera and Nicolini (2004) show that a unique constant maturity structure $\boldsymbol{b}^*$ supports the Ramsey tax plan in this economy. A more general statement can be made regarding the entire set of feasible tax systems with rates contingent only on the exogenous state.

Proposition 2. *For all feasible tax systems contingent only on the exogenous state, there exists a unique invariant supporting maturity structure.*

Equation (16) says that the state-contingent transfers from the government to households must be consistent with payouts on the government's portfolio of debt along the Ramsey tax plan.

This proposition simply defines Z to be an arbitrary feasible set of transfers contingent on the exogenous states g. And defines A to be a matrix of payouts consistent with the feasible plan. So long as A is non-singular (which is satisfied through $S = I$ and the concavity and convexity properties of utility in consumption and labor, respectively), there is a unique mapping from Z to construct $\boldsymbol{b}$ akin to (16). The proof can be found in appendix A.

4.6 Debt Return-Targeting Debt-Manager, Welfare-Maximizing Central Bank

It is natural to start the analysis with a welfare-maximizing central bank:

$$V_{cb} = \mathbb{E} \sum_{t}^{\infty} \beta^t u(c_t, n_t)$$

The central bank solves the Ramsey problem while taking the debt manager's choice $\hat{b}_t$ as an additional constraint.[11]

According to the US Treasury, official national debt management policy is to achieve the "least

[11] Timing assumptions affect the nature of this constraint. If the central bank is either in a simultaneous game or moves second in a Stackelberg setting, this constraint is correct. If the central bank is a first-mover, it instead takes the debt-manager's best response function as a constraint.

expected cost over time" and is to be "regular and predictable."[12] Define asset j's ex-post net rate of return at time t as the $r_t^{(t+j)}$ that satisfies:

$$1 + r_t^{(t+j)} = \frac{q_t^{(t+j)}}{q_{t-1}^{(t+j)}} \tag{17}$$

Model the debt-manager as one that aims to achieve a target total debt payment in every period:

$$V_{dm} = -\mathbb{E}\sum_{t}^{\infty} \beta^t \left(k_t - \sum_{j=0}^{J} r_t^{(t+j)} \hat{b}_{t-1}^{(t+j)} \right)^2$$

where k_t is the debt payment target for time t, chosen at $t-1$.

To keep the model solution stable, restrict these target choices to vary contingently on both the state in which they're chosen at $t-1$ as well as the ex-post realized state at period t. The debt-service target chosen by the $t-1$ debt manager in state s for when state s' is realized at time t is written as:

$$k_t = k_{t,s'}(s)$$

and further define the entire vector of these targets as $K_t(s)$ so that:

$$K_t(s) = \begin{bmatrix} k_{t,1}(s) & k_{t,2}(s) & \cdots & k_{t,S}(s) \end{bmatrix}'$$

4.7 Non-Cooperation under Effectively Complete Markets

In a simultaneous-move game, the Ramsey tax plan is achieved by Proposition 1 as the central bank is welfare-maximizing.[13] So both the debt payouts and the outstanding maturity structure is identical to that in the Ramsey problem. Here the central bank optimally adjusts $\boldsymbol{b_t}$ in every period such that $\boldsymbol{b_t} = \boldsymbol{b}^*$ through (8).

[12] Quotes are from a **Nov. 2020 US Treasury memo**.
[13] The welfare-maximizing equilibrium also exists when the central bank both moves first and moves second. It is unique with a central bank follower. Under a central bank leader, the welfare-maximizing equilibrium is selected if the debt-manager places even an infinitesimal amount of weight on welfare.

Define a state-contingent matrix of *ex-post* net interest rates along the Ramsey plan as $R(s)^*$ so that:

$$R(s)^* \equiv \begin{bmatrix} \frac{1}{q_{t-1}^{(t)}(s)^*} & \frac{q_t^{(t+j_2-1)}(1)^*}{q_{t-1}^{(t+j_2-1)}(s)^*} & \cdots & \frac{q_t^{(t+J-1)}(1)^*}{q_{t-1}^{(t+J-1)}(s)^*} \\ \frac{1}{q_{t-1}^{(t)}(s)^*} & \frac{q_t^{(t+j_2-1)}(2)^*}{q_{t-1}^{(t+j_2-1)}(s)^*} & \cdots & \frac{q_t^{(t+J-1)}(2)^*}{q_{t-1}^{(t+J-1)}(s)^*} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{q_{t-1}^{(t)}(s)^*} & \frac{q_t^{(t+j_2-1)}(S)^*}{q_{t-1}^{(t+j_2-1)}(s)^*} & \cdots & \frac{q_t^{(t+J-1)}(S)^*}{q_{t-1}^{(t+J-1)}(s)^*} \end{bmatrix} - \mathbb{O}_I \quad \forall s \in \{1, \ldots, S\}$$

where $\mathbb{O}_I$ is a $I \times I$ matrix of ones.

Define $Q(s)^*$ as a diagonal matrix of inverse $t-1$ debt prices at each maturity:

$$Q(s)^* \equiv \begin{bmatrix} \frac{1}{q_{t-1}^{(t)}(s)^*} & 0 & \cdots & 0 \\ 0 & \frac{1}{q_{t-1}^{(t+1)}(s)^*} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{q_{t-1}^{(t+J-1)}(s)^*} \end{bmatrix}$$

so that $R(s)^*$ may be rewritten as a function of the invariant payout matrix A^* and $Q(s)^*$:

$$R(s)^* = A^* Q(s)^* - \mathbb{O}_J$$

By definition, the debt-manager's debt service payments along the Ramsey plan at time t evolve according to:

$$K_t(s) = K(s) = R(s)^* \, \hat{b}_{t-1}^* \tag{18}$$

and a state-contingent, optimal structure of debt held by the debt-manager is calculated as:

$$\hat{b}_{t-1}^* = \hat{b}(s)^* = R(s)^{*^{-1}} K(s) \tag{19}$$

where $\{\hat{b}(s)^*\}_s$ maximizes V_{dm} at $V_{dm} = 0$ along the Ramsey plan. So that for any $\{K(s)\}_s \in \mathbb{R}^S \times \mathbb{R}^S$, there exists a set of exogenous-state-contingent debt-manager debt choices such that the debt-manager perfectly achieves its objective. $V_{dm} = 0$ is achieved along any feasible tax plan that satisfies $\tau_t \in \{\tau(s)\}\ \forall t$, making it a more general result than the case examined here.

Regardless of the debt-manager's choice of debt issuance, the resulting behavior of the central bank is to maximize welfare by simply setting its debt-holding strategy as:

$$\tilde{b}_t^* = \tilde{b}(s)^* = \hat{b}(s)^* - \boldsymbol{b}^*$$

or:

$$\tilde{b}(s)^* = R(s)^{*^{-1}} K(s) - A^{*^{-1}} Z^* \tag{20}$$

All institutions perfectly satisfy their individual objectives. Household welfare is maximized as the conditions for Proposition 1 are met. The SBNE is efficient.

One can view the debt-manager as the 'dominant' authority in this equilibrium. The debt-manager is allowed to pursue a non-welfare-maximizing objective without consequence as the central bank cleans up the the mess. But the central bank is also constraining the debt-manager - the central bank's debt response affects tax policy through the solvency condition. Tax policy regulates the realized interest rate schedule, feeding back to the debt-manager's issuance and redemption choices. From the viewpoint of either institution, the other is complicating their problem.

4.7.1 Special Case of Debt-Manager Issuance Behavior

In many models, the debt-manager is assumed to follow an issuance rule. These rules can map into this model's framework.

A special case arises when the debt-manager aims to have the portfolio's return equal a weighted sum of specified asset returns. In this case, $\hat{b}(s) = \gamma \left[\gamma^{(1)}\ \gamma^{(2)}\ \cdots\ \gamma^{(I)}\right]'$ where $\gamma^{(1)} + \gamma^{(2)} + \cdots + \gamma^{(I)} = 1$ and where γ is a constant.

A debt-manager chooses $\hat{b}(s)^* = \gamma\,[1\ 1\ \cdots\ 1]'$ (a more specific version of the above) when it targets state-contingent debt service as a multiple of the (unweighted) mean of next period's interest rates across the portfolio. Models with a flat issuance strategy from the debt-manager are consistent with this objective specification.

When debt service targets are tied to the future yield curve as in these cases, the debt-manager's issuance strategy becomes time-invariant (so that $\hat{b}(s)^* = \hat{b}^*$).

4.8 An Example with Two States and Two Maturities

Consider a version of the model where $S = I = 2$. Call government spending 'low' when $s = 1 = \ell$ and 'high' when $s = 2 = h$ so that $g(s) \in \{g(\ell), g(h)\}$ and $g(\ell) < g(h)$. The maturity structures for each institution are solvable analytically.[14]

The planner's choice of short-term debt equates the ratio of state-contingent long-term debt payouts to the ratio of maturing short-term debt's shortfalls from covering optimal state-contingent transfers to households. It chooses $\boldsymbol{b}_t^{(t+1)^*}$ $\forall t$ such that the following holds:

$$\underbrace{\frac{q_{t+1}^{(t+J)}(\ell)^*}{q_{t+1}^{(t+J)}(h)^*}}_{\text{ratio of LT debt payouts}} = \underbrace{\frac{\boldsymbol{b}_t^{(t+1)^*} - z(\ell)^*}{\boldsymbol{b}_t^{(t+1)^*} - z(h)^*}}_{\text{ratio of transfer shortfalls from ST debt}} \tag{21}$$

When the economy enters period $t+1$ after exiting t, the payout of J-period debt to bondholders is the price of $J-1$-period debt. The payout of one-period debt is 1 as maturing debt is non-contingent.

The price of $J-1$-period debt in state s at time $t+1$ can be thought of the ex-post 'value' of saving in two period debt in period t. The more valuable long-term debt is in state ℓ relative to state h, the more willing the planner is to set one-period debt far away from state ℓ's Ramsey transfer relative to state h's Ramsey transfer.

The planner chooses J-period debt to simply satisfy the GBC in period $t+1$ upon realizing state s. It chooses $\boldsymbol{b}_t^{(t+J)^*}$ $\forall t$ such that the following holds:

$$\underbrace{z(h)^* - q_{t+1}^{(t+J)}(h)^*\boldsymbol{b}_t^{(t+J)^*}}_{\text{maturing } \boldsymbol{b}_t^{(t+1)^*} \text{ in high state}} = \underbrace{z(\ell)^* - q_{t+1}^{(t+J)}(\ell)^*\boldsymbol{b}_t^{(t+J)^*}}_{\text{maturing } \boldsymbol{b}_t^{(t+1)^*} \text{ in low state}} \tag{22}$$

Equations (21) and (22) characterize a perfect hedging (perfect insurance) solution for the planner. Following this solution ensures the economy stays on the Ramsey plan.

[14] Appendix C goes through these derivations.

4.8.1 The Closed-Form Ramsey Planner's Solution

From equations (21) and (22), solve for the optimal maturity structure $\boldsymbol{b}_t^* = \boldsymbol{b}^*\ \forall t$:

$$\boldsymbol{b}_t^{(t+1)^*} = \left(\frac{q_{t+1}^{(t+J)}(h)^*}{q_{t+1}^{(t+J)}(h)^* - q_{t+1}^{(t+J)}(\ell)^*} \right) z(\ell)^* - \left(\frac{q_{t+1}^{(t+J)}(\ell)^*}{q_{t+1}^{(t+J)}(h)^* - q_{t+1}^{(t+J)}(\ell)^*} \right) z(h)^*$$

$$\boldsymbol{b}_t^{(t+J)^*} = - \left(\frac{1}{q_{t+1}^{(t+J)}(h)^* - q_{t+1}^{(t+J)}(\ell)^*} \right) z(\ell)^* + \left(\frac{1}{q_{t+1}^{(t+J)}(h)^* - q_{t+1}^{(t+J)}(\ell)^*} \right) z(h)^*$$

4.8.2 The Closed-Form Service-Targeting Debt-Manager's Solution

When $s = h$, the debt-manager's issuance choice satisfies:

$$\underbrace{\frac{q_t^{(t+J-1)}(\ell)^*}{q_t^{(t+J-1)}(h)^*}}_{\text{ratio of LT debt payouts}} = \frac{\overbrace{r_t^{(t)}(h)^* \hat{b}_{t-1}^{(t)^*} - k_{t,\ell}(h)}^{\text{1-period return minus next period target if } \ell \text{ realized}} + \overbrace{\mathbb{E}_h \left[q_t^{(t+1)}(s)^* \right] \left(k_{t,\ell}(h) - k_{t,h}(h) \right)}^{\text{forward weighted difference in targets}}}{\underbrace{r_t^{(t)}(h)^* \hat{b}_{t-1}^{(t)^*} - k_{t,h}(h)}_{\text{1-period return minus next period target if } h \text{ realized}}}$$

When $s = \ell$, it satisfies:

$$\frac{q_t^{(t+1)}(\ell)^*}{q_t^{(t+1)}(h)^*} = \frac{r_t^{(t)}(\ell)^* \hat{b}_{t-1}^{(t)^*} - k_{t,\ell}(\ell)}{r_t^{(t)}(\ell)^* \hat{b}_{t-1}^{(t)^*} - k_{t,h}(\ell) + \mathbb{E}_\ell \left[q_t^{(t+1)}(s)^* \right] \left(k_{t,h}(\ell) - k_{t,\ell}(\ell) \right)}$$

Using these equations, the debt-manager's issuance choice can be written as a linear combination of the debt service targets:

$$\hat{b}_t^{(t+1)}(s)^* = \left(\frac{q_t^{(t+1)}(s)^* \left(q_{t+1}^{(t+J)}(h)^* - q_t^{(t+J)}(s)^* \right)}{\left(1 - q_t^{(t+1)}(s)^*\right) \left(q_{t+1}^{(t+J)}(h)^* - q_{t+1}^{(t+J)}(\ell)^* \right)} \right) k_{t+1,\ell}(s) - \left(\frac{q_t^{(t+1)}(s)^* \left(q_{t+1}^{(t+J)}(\ell)^* - q_t^{(t+J)}(s)^* \right)}{\left(1 - q_t^{(t+1)}(s)^*\right) \left(q_{t+1}^{(t+J)}(h)^* - q_{t+1}^{(t+J)}(\ell)^* \right)} \right) k_{t+1,h}(s)$$

$$\hat{b}_t^{(t+J)}(s)^* = - \left(\frac{q_t^{(t+J)}(s)^*}{q_{t+1}^{(t+J)}(h)^* - q_{t+1}^{(t+J)}(\ell)^*} \right) k_{t+1,\ell}(s) + \left(\frac{q_t^{(t+J)}(s)^*}{q_{t+1}^{(t+J)}(h)^* - q_{t+1}^{(t+J)}(\ell)^*} \right) k_{t+1,h}(s)$$

Following this strategy ensures that the debt manager perfectly hits its targets in every period, regardless of how the economy evolves.

4.8.3 The Closed-Form Welfare-Maximizing Central Bank's Solution

And the welfare-maximizing central bank's choice is simply holding $\tilde{b}(s)^*$ on its balance sheet such that $\tilde{b}(s)^* = \hat{b}(s)^* - \boldsymbol{b}^*$ in every period:

$$\tilde{b}_t^{(t+1)}(s)^* = \left(\frac{q_t^{(t+1)}(s)^* \left(q_{t+1}^{(t+J)}(h)^* - q_t^{(t+J)}(s)^* \right)}{\left(1 - q_t^{(t+1)}(s)^*\right)\left(q_{t+1}^{(t+J)}(h)^* - q_{t+1}^{(t+J)}(\ell)^* \right)} \right) k_{t+1,\ell}(s) - \left(\frac{q_t^{(t+1)}(s)^* \left(q_{t+1}^{(t+J)}(\ell)^* - q_t^{(t+J)}(s)^* \right)}{\left(1 - q_t^{(t+1)}(s)^*\right)\left(q_{t+1}^{(t+J)}(h)^* - q_{t+1}^{(t+J)}(\ell)^* \right)} \right) k_{t+1,h}(s)$$

$$- \left(\frac{q_{t+1}^{(t+J)}(h)^*}{q_{t+1}^{(t+J)}(h)^* - q_{t+1}^{(t+J)}(\ell)^*} \right) z(\ell)^* + \left(\frac{q_{t+1}^{(t+J)}(\ell)^*}{q_{t+1}^{(t+J)}(h)^* - q_{t+1}^{(t+J)}(\ell)^*} \right) z(h)^*$$

$$\tilde{b}_t^{(t+2)}(s)^* = - \left(\frac{q_t^{(t+J)}(s)^*}{q_{t+1}^{(t+J)}(h)^* - q_{t+1}^{(t+J)}(\ell)^*} \right) k_{t+1,\ell}(s) + \left(\frac{q_t^{(t+J)}(s)^*}{q_{t+1}^{(t+J)}(h)^* - q_{t+1}^{(t+J)}(\ell)^*} \right) k_{t+1,h}(s)$$

$$+ \left(\frac{1}{q_{t+1}^{(t+J)}(h)^* - q_{t+1}^{(t+J)}(\ell)^*} \right) z(\ell)^* - \left(\frac{1}{q_{t+1}^{(t+J)}(h)^* - q_{t+1}^{(t+J)}(\ell)^*} \right) z(h)^*$$

5 The Model Under Incomplete Markets

5.1 A Central Bank Unable to Issue Long-Term Debt

One possible implication of the optimal central bank response under effectively complete markets is that at least one long-term debt position in $\left\{ \left\{ \tilde{b}_t^{(t+j)}(s)^* \right\}_{j>1} \right\}_s$ may be negative. Or that the central bank holds long-term liability positions on its balance sheet (issues its own long-term debt).

In the United States, the Fed's long-term liabilities are constrained to be non-negative.[15] Impose the following set of short-selling constraints on the model:

$$\tilde{b}_t^{(t+j)} \geq 0 \quad j \in \{j_2, \ldots, J\} \tag{23}$$

[15]The constraint is imposed by law. The Fed has no right to issue long dated liabilities. Marketable liabilities on the Fed's balance sheet include bank, Treasury, foreign and other deposits, outstanding reserve notes and reverse repos, which all are classified as short-term.

No additional constraint is placed on the debt-manager's issuance/redemption behavior. It's important to note that the consolidated budget constraint continues to hold in all states under this constraint.

Equation (23) moves any version of the model where at least one element of $\left\{\left\{\tilde{b}_t^{(t+j)}(s)^*\right\}_{j>1}\right\}_s$ is negative from a complete markets model to one where markets are incomplete. Assume that this condition holds for the remainder of the analysis.[16] Following Aiyagari et al. (2002), the state space under incomplete markets bloats to include lagged household Lagrange multipliers and consolidated debt holding from $t-1$ to $t-J$. Call X_t the state vector at time t:

$$X_t = \left\{g_t, \{\lambda_{t-i}\}_{i=1}^{J}, \{\boldsymbol{b}_{t-i}\}_{i=1}^{J}\right\}$$

which means that for a seemingly innocuous $I=2$, $J=15$ specification, the state vector contains 46 elements.[17]

Under a simultaneous-mover set-up, these additional constraints on the model limit the central bank's strategy space so that $\Gamma_{cb}\left(X_t|\hat{b}_t\right) = \mathbb{R} \times \mathbb{R}_+^{I-1}$. The debt-manager's strategy space is unchanged: $\Gamma_{dm}\left(X_t|\tilde{b}_t\right) = \mathbb{R}^I$ continues to hold.

Without either a welfare-maximizing debt manager or a central bank whose strategy space is $\Gamma = \mathbb{R}^I$, Proposition 1 breaks down. The allocation resulting from the decentralized game is no longer guaranteed match that chosen by the planner.[18]

5.1.1 Simultaneous Debt-Manager and Central Bank Movement

First examine a simultaneous game between the debt-manager and central bank. Each institution takes the other's choice allocation as given (as opposed to optimizing over the entire opposing institution's best response function).[19]

One criteria for a SPNE is that all players must be content with their choice of strategy, even given other player's choice. As the debt-manager's strategy space is unchanged from section 4, it

[16] If no short-selling constraint binds in any state, the model continues to have effectively complete markets. The model requires extreme assumptions of the debt-manager's service targets for this to occur when calibrated in the next section.

[17] $I=2$, $J=15$ is part of the calibration later in the paper.

[18] The additional assumption that the central bank's complete markets long-term position is negative in at least one state guarantees that these allocations are not identical.

[19] Though best response functions are used to form expectations of future institutional decision-making.

continues to maximize its objective function by following equation (19). It does not ensure $V_{dm} = 0$ with any other feasible strategy.

This result simplifies the game because it equates the game's problem to that of a welfare-maximizing central bank who takes the debt-manager's choice as given. The central bank additionally faces the consolidated GBC, the ARC and HH optimization as constraints. Finally, it faces an additional $I - 1$ short-selling constraints that may not bind.[20] Assume that the central bank has commitment power to future policy.

Combine the time-t consolidated GBC, ARC, HH optimization to arrive at a measurability constraint:

$$\sum_{j=1}^{J} \beta^j \mathbb{E}_t \left[c_{t+j}^{-\sigma}\right] \boldsymbol{b}_t^{(t+j)} - \sum_{j=1}^{J} \beta^{j-1} \mathbb{E}_t \left[c_{t+j-1}^{-\sigma}\right] \boldsymbol{b}_{t-1}^{(t+j-1)} = (c_t + g_t)^{1+\varphi} - c_t^{1-\sigma} \tag{24}$$

And the first order conditions are:

$$c_t: \quad c_t^{-\sigma} - (c_t + g_t)^{\varphi} + \lambda_t \left[(1-\sigma)\, c_t^{-\sigma} - (1-\varphi)\,(c_t + g_t)^{\varphi}\right] - \sigma c_t^{-(1-\sigma)} \sum_{j=1}^{J} (\lambda_{t-j} - \lambda_{t-j+1})\, \boldsymbol{b}_{t-j}^{(t)} = 0 \tag{25}$$

$$\boldsymbol{b}_t^{(t+j)}: \quad \lambda_t = \mathbb{E}_t \left[c_{t+j}^{-\sigma}\right]^{-1} \left(\mathbb{E}_t \left[\lambda_{t+1} c_{t+j}^{-\sigma}\right] + \beta^{-j} \mu_t^{(t+j)}\right) \quad \forall\, j \in \{1, \ldots, J\} \tag{26}$$

where $\mu_t^{(t+j)}$ is the Lagrange multiplier on the short-selling constraints and where $\boldsymbol{b}^{(t+j)} = \hat{b}_t^{(t+j)} - \tilde{b}_t^{(t+j)}$.[21] The central bank takes $\hat{b}_t$ as given during the simultaneous-move game, so choosing $\boldsymbol{b}^{(t+j)}$ is isomorphic to choosing its native choice variable $\tilde{b}_t^{(t+j)}$.

These conditions are similar to those found in most papers that study optimal debt management with incomplete markets, including Barro (1979), Aiyagari et al. (2002) and Valaitis and Villa (2023).

The Lagrange multiplier takes the form of a random walk. Though, as will be shown in the calibrated model's solution, the multiplier (along with most objects in the model) remains fairly stable due to the precise nature of the market incompleteness introduced by the short-selling constraints.

In order to simulate the model, solve the $I + 2$-equation system of (24), (25) and (26) for λ_t, c_t

[20] Short-selling constraints can be rewritten as $\hat{b}_t^{(t+j)} \geq \boldsymbol{b}_t^{(t+j)}\ \forall\, j > 1$ so that the constraints serve as upper bounds on the consolidated governments long-term debt holding.

[21] $\mu_t^{(t+1)} = 0$ as the short-selling constraint only applies to long-term debt $j > 1$.

and $\boldsymbol{b}_t$ in each period. More details on solving the model can be found in section 6.3.

5.1.2 A Debt-Manager Leader, Central Bank Follower

When the debt-manager leads, it takes the central bank's best response function into account when optimizing while the central bank continues to take $\hat{b}_t$ as given every period. Neither the central bank's maximization problem nor it's optimality conditions change.

The debt-manager achieves its objective function's global maximum $V_{dm} = 0$ when it follows equation (19). As the debt prices in equation (19) are driven by the welfare-maximizing central bank, the debt-manager first-mover Stackelberg equilibrium is identical to the simultaneous-move SPNE. There is no first- or second-mover advantage when the debt-manager moves first.

Because this is a model with complete information and a SPNE exists in the simultaneous-move game, no second-mover advantage can arise. Therefore, any advantage will originate from moving first. The first-mover advantage arises from constraining the opponent's choice to make the SPNE more favorable for the first mover. The concept of a first-mover advantage does not exist in the debt-manager's case because it can't do better than the simultaneous-mover specification.

Both the central bank's and debt-manager's solutions are identical to the simultaneous-mover case. So too will be the allocations.

5.1.3 A Central Bank Leader, Debt-Manager Follower

The central bank enjoys a first-mover advantage as, rather than taking $\hat{b}_t$ as given, it's FOCs include optimizing over the debt-manager's best response function (19). It moves the debt-manager from the simultaneous-move allocation to one with slightly higher welfare, closer to the Ramsey solution. Understanding how the debt-manager's choice $\hat{b}_t$ is affected by $\tilde{b}_t$ yields a an additional payout for the central bank, which results in welfare gain relative to the SPNE.

While the debt-manager's allocation is different from the allocations in the previous two examples, it still achieves $V_{dm} = 0$ by following (19) because debt prices are still well-defined under the new optimization. A central bank leader is best for society because of its ability to constrain the debt-manager (even while the debt-manager is still able to perfectly target debt returns).

6 Calibrating and Solving the Model

6.1 Identification

Calibrate a two-state, two-maturity model to match moments in yearly US data from 1942-2022. Data is taken from the Buereau of Economic Analysis's National Income and Product Accounts (NIPA) tables and Hall and Sargent (2011).

To calibrate the spending amounts $\{g(s)\}$ that the government switches between, use yearly data on US (spending+transfers)/GDP. Calling the data $\{x_t\}$, two spending values $(g_1, g_2) = (.1751, .357)$ solve:

$$\underset{g_1, g_2}{\operatorname{argmin}} \sum_{t=1942}^{2022} \min\left[(g_1 - x_t)^2 \;,\; (g_2 - x_t)^2\right]$$

Transition matrix probabilities are calibrated to match the frequency and average duration of spending switching in the data, shown in Table 1. The model's probability matrix matches the data according to Table 2. A full description of the model calibration is shown in Table 3.

Data	$g_1 = .1751$	$g_2 = .357$
$Pr(g_s)$	.925	.075
Avg. duration (years)	74	3

Table 1: State Transition Data

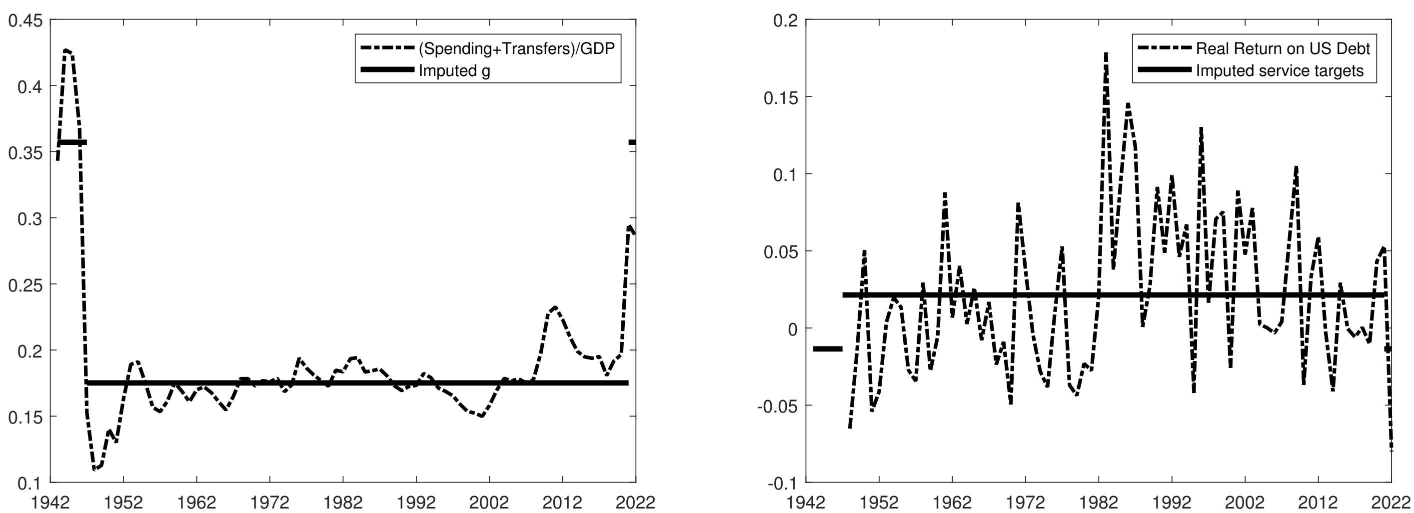
Model	$g_1 = .1751$	$g_2 = .357$
$Pr(g_s)$	.9268	.0732
Avg. duration (periods)	64.5	5.26

Table 2: State Transition Matching

Variable	Calibration	Description
(g_1, g_2)	$(.1751, .3570)$	Distance-minimizing amounts
(p_{11}, p_{12})	$(.985, .015)$	Transition matrix probabilities
(p_{21}, p_{22})	$(.19, .81)$	used to match state-switching dynamics in data
β	$1/1.05$	5% annual rate
σ	2	Conventional estimate of 1/IES
φ	2	Conventional estimate of 1/FrischELS
$K(s)\ \forall s$	$(.02141, -.013536)$	Debt return targets (matching real returns on debt in data)
$\{j\}$	$(1, 15)$	Maturity lengths (matching US notes, avg(bills, bonds) lengths)

Table 3: Model Calibration

Figure 1's first plot outlines identification of the model's exogenous spending calibration. The second plot shows real holding period returns on the government debt portfolio. State-contingent averages of debt service (returns on government debt) are used in the calibration of $K = [k(s) \quad k(s)]$. Debt service targets are calibrated to only be state contingent on the state at t and not the one at $t-1$ (although the theory permits it).



(a) US Spending and Transfers

(b) Treasury Debt Service

Figure 1: State Switching in the US Economy

Because the model is calibrated on an annual basis and the US debt structure includes up to

30-year debt, debt maturity lengths of 1 and 15 years are used to match short-term (notes) and the average maturity of medium-term (bills) and long-term (bonds) debt. All other calibrations (β, σ, φ) follows standard estimates. Calculations are for an economy beginning in the low-spending regime.

6.2 Solving the Model Under Effectively Complete Markets

The model solution under effectively complete markets when $I = S = 2$, $J = 15$ is straightforward once the Ramsey planner's problem is worked out.[22] The objective is to numerically solve for the constant $\tilde{\lambda}$ that satisfies the following system:

$$\begin{cases} (c_t + g_t)^{\varphi} = \tilde{\lambda} c_t^{-\sigma} \\ 0 = \mathbb{E} \sum_t^{\infty} \beta^t \left[(c_t)^{1-\sigma} - (c_t + g_t)^{1+\varphi} \right] \end{cases}$$

where the first equation comes from household optimization and the second one is the government's implementability constraint.

Do this by defining $c_t = c\left(\tilde{\lambda}, g_t\right)$ as the solution to the first equation in the system and substitute it into the second to write:

$$0 = \mathbb{E} \sum_t^{\infty} \beta^t \left[\left(c\left(\tilde{\lambda}, g_t\right) \right)^{1-\sigma} - \left(c\left(\tilde{\lambda}, g_t\right) + g_t \right)^{1+\varphi} \right]$$

which can be simplified more by including that $g_t = g(s) \implies c_t = c(s) \quad \forall\, t$:

$$0 = \mathbb{E}_s \sum_t^{\infty} \beta^t \left[\left(c\left(\tilde{\lambda}, g(s)\right) \right)^{1-\sigma} - \left(c\left(\tilde{\lambda}, g(s)\right) + g(s) \right)^{1+\varphi} \right]$$

Choosing the constant $\tilde{\lambda}$ that minimizes the square of the RHS given the exogenous states $\{g(s)\}$, the transition matrix P, the starting state and the expectations operator $\mathbb{E}$ gives us $\tilde{\lambda}^*$, the Ramsey planner's optimal $\tilde{\lambda}$.

Use $\tilde{\lambda}^*$ to solve for $\{c(s)^*\}_s$, $\{n(s)^*\}_s$ and τ^*. Then use the planner's allocations with the decentralized model equations to back out $\{q(s)^*\}_s$, $\boldsymbol{b}^*$, $\{\hat{b}(s)^*\}_s$, $\{\tilde{b}(s)^*\}_s$, $\{\{r^*_{s_{t-1},s_t}\}_{s_{t-1}}\}_{s_t}$ and $\{\{\Delta^*_{s_{t-1},s_t}\}_{s_{t-1}}\}_{s_t}$.

[22] Appendix B goes through the solution.

6.3 Solving the Model Under Incomplete Markets

This section outlines the approximation method used to solve the model from section 5. The simulated neural network (SNN) approach is similar to the simulated parameterized expectations algorithm (SPEA). The goal here is to solve the model over a long simulation of its exogenous process. The neural network iteratively trains after each simulation on that simulation's generated data. The algorithm is later discussed in detail.

Seen first in section 3.3 of Valaitis and Villa (2023), SNN relies on neural networks to generate expectations. Neural networks are better adept at handling multicolinearity generated by highly-substitutable debt objects than other common solution methods making it an ideal choice for this analysis.[23]

The state space X_t is a $1 + J(1 + I)$ vector of lagged debt holdings and Lagrange multipliers:

$$X_t = \left\{ g_t, \{\lambda_{t-i}\}_{i=1}^{J}, \{\boldsymbol{b}_{t-i}\}_{i=1}^{J} \right\}$$

Given X_t, a trained neural network approximates $3 \times I$ expectations. Call the neural network $\mathcal{NN}$ a mapping from the state X_t to a set of expectations:

$$\begin{aligned} \mathcal{NN}_1^{(j)}(X_t) &= \mathbb{E}_t\left[c_{t+j}^{-\sigma}\right] && \text{for } j = \{1, \cdots, J\} \\ \mathcal{NN}_2^{(j)}(X_t) &= \mathbb{E}_t\left[\lambda_{t+j} c_{t+j}^{-\sigma}\right] && \text{for } j = \{1, \cdots, J\} \\ \mathcal{NN}_3^{(j)}(X_t) &= \mathbb{E}_t\left[c_{t+j-1}^{-\sigma}\right] && \text{for } j = \{1, \cdots, J\} \end{aligned}$$

In the calibrated model where $I = S = 2$ and $J = 15$, this results in 5 approximations (because short-term debt has a maturity of 1, $\mathcal{NN}_3^{(1)}$ is simply $c_t^{-\sigma}$).

Solution Algorithm: The approach is used to solve the $I = 2$, $J = 15$ model as a welfare-maximizing consolidated government with an additional constraint: $\hat{b}_t^{(t+J)} \geq \boldsymbol{b}_t^{(t+J)}$. The debt-manager's decision is solved using debt prices at each period so that the central bank's choices can be solved using market clearing.

1. Begin by adding a set of additional bounds on the consolidated government. Namely, upper

[23] This comparative advantage is the main point of Valaitis and Villa (2023).

and lower bounds on short-term, long-term and total debt. These bounds are called Maliar bounds after their implementation in Maliar and Maliar (2003). Adding Maliar bounds results in a rewriting of the government's FOC for debt (26):

$$\lambda_t = \mathcal{NN}_1^{(j)}\left(X_t\right)^{-1}\left(\mathcal{NN}_2^{(j)}\left(X_t\right) + \beta^{-j}\left(\mu_t^{(j)} + \zeta_{U,t}^{(j)} - \zeta_{L,t}^{(j)} + \zeta_{U,t}^{\text{Total}} - \zeta_{L,t}^{\text{Total}}\right)\right) \quad \forall\ j \in \{1, J\}$$

These bounds need to begin tightly around zero and slowly relax as the number of simulations increase. Using penalty functions to replace the μ_t and ζ_ts increases accuracy and does a better job of keeping the simulation under control.

2. When the maturity structure of public debt includes more than one maturity, the above equation overidentifies λ_t.[24] Only two equations (measurability constraint and consumption FOC) are left to solve for three variables in each period: c_t, $b_t^{(t+1)}$ and $b_t^{(t+J)}$. Following Faraglia et al. (2019), the forward-states approach is used to solve this issue. The idea is to approximate current expected-value allocations with X_t and use the law of iterated expectations to calculate their future expectations. For instance, $\mathbb{E}_t\left[\mathcal{NN}_1^{(j)}\left(X_{t+1}\right)\right] = \mathbb{E}_t\left[c_{t+j}^{-\sigma}\right]$ rather than $\mathcal{NN}_1^{(j)}\left(X_t\right) = \mathbb{E}_t\left[c_{t+j}^{-\sigma}\right]$.[25]

3. The simulation involves repeatedly solving the following four-equation system to find $\{c_t\}_{t=0}^T$, $\{\lambda_t\}_{t=0}^T$, $\{\boldsymbol{b}_t^{(t+1)}\}_{t=0}^T$ and $\{\boldsymbol{b}_t^{(t+J)}\}_{t=0}^T$:

$$\begin{cases} \lambda_t = \mathbb{E}_t\left[\mathcal{NN}_1^{(1)}\left(X_{t+1}\right)\right]^{-1}\left(\mathbb{E}_t\left[\mathcal{NN}_2^{(1)}\left(X_{t+1}\right)\right] + \beta^{-1}\left(\zeta_{U,t}^{(1)} - \zeta_{L,t}^{(1)} + \zeta_{U,t}^{\text{Total}} - \zeta_{L,t}^{\text{Total}}\right)\right) \\ \lambda_t = \mathbb{E}_t\left[\mathcal{NN}_1^{(J)}\left(X_{t+1}\right)\right]^{-1}\left(\mathbb{E}_t\left[\mathcal{NN}_2^{(J)}\left(X_{t+1}\right)\right] + \beta^{-J}\left(\mu_t^{(J)} + \zeta_{U,t}^{(J)} - \zeta_{L,t}^{(J)} + \zeta_{U,t}^{\text{Total}} - \zeta_{L,t}^{\text{Total}}\right)\right) \\ c_t^{-\sigma} - \left(c_t + g_t\right)^{\varphi} + \lambda_t\left[\left(1-\sigma\right)c_t^{-\sigma} - \left(1-\varphi\right)\left(c_t + g_t\right)^{\varphi}\right] = \sigma c_t^{-(1-\sigma)}\left(\left(\lambda_{t-1} - \lambda_t\right)\boldsymbol{b}_{t-1}^{(t)} + \left(\lambda_{t-J} - \lambda_{t-J+1}\right)\boldsymbol{b}_{t-J}^{(t)}\right) \\ \beta\mathbb{E}_t\left[\mathcal{NN}_1^{(1)}\left(X_{t+1}\right)\right]\boldsymbol{b}_t^{(t+1)} + \beta^J\mathbb{E}_t\left[\mathcal{NN}_1^{(J)}\left(X_{t+1}\right)\right]\boldsymbol{b}_t^{(t+J)} = \\ \qquad c_t^{-\sigma}\boldsymbol{b}_{t-1}^{(t)} + \beta^{J-1}\mathbb{E}_t\left[\mathcal{NN}_3^{(J)}\left(X_{t+1}\right)\right]\boldsymbol{b}_{t-1}^{(t+J-1)} + \left(c_t + g_t\right)^{1+\varphi} - c_t^{1-\sigma} \end{cases} \tag{27}$$

where T is a number large enough to reliably train the neural network and small enough to

[24] The RHS only consists of predetermined variables.

[25] Faraglia et al. (2019) discusses the non-singularity problem and forward states approach more thoroughly.

keep the computation time reasonable.

4. If the average approximation error of the system is too large at simulation i, use the $i-1$'s neural network and run the algorithm with a smaller Maliar bound.

5. If the solution from step 4 is still not sufficient, continue to use $i-1$'s neural network but further reduce the bound somewhere between $Bound_{i-1}$ and $Bound_{i-2}$. Continue to do this until the simulation completes without large errors. At that point, calculate the simulation with $Bound_i = Bound_{i-1} + (Bound_{i-1} - Bound_{i-2})$. This ensures that the algorithm continues through local error maxima.

6. If simulation i's errors are small, the neural network trains on simulation i's data (after some burn-in period to tease out effects from the model's initial conditions) to prepare itself to approximate terms in simulation $i+1$. Increase the Maliar bounds and begin simulation $i+1$.

Continue this algorithm until the neural network, $\{c_t\}$ and $\{\boldsymbol{b}_t\}$ remain constant from simulation $i-1$ to i. The process is discussed further in Valaitis and Villa (2023).

7 Model Results

7.1 Effectively Complete Markets Results

Table 4 lists model results under the effectively complete markets specification outlined in section 4 when calibrated according to section 6.1.

These results are consistent with established findings in the literature along three dimensions. First, *perfectly smoothing tax rates according to the Ramsey plan is optimal and requires large absolute positions in outstanding debt.* This is due to the high degree of correlation of yields across maturities upon realizing a new state. In order to construct a risk-minimizing portfolio of highly similar assets, large offsetting positions are taken.[26] Second, *the yield curve inverts in periods of high government spending.* Government debt is more valuable in low-spending states when

[26]These offsetting positions arise largely due to how asset prices move from state to state. Bhandari et al. (2017) investigates how the optimal maturity structure differs when asset prices are more consistent with reality. Their results differ greatly.

marginal utility is low. Upon moving to a high-spending state, government consumption crowds out private consumption which increases marginal utility. Debt prices (and thus, ex-post returns) fall as agents move away from saving. Third, *a maturity structure consisting of short-term assets and long-term debt is required by the government when implementing the first-best allocation.* The government optimally constructs its portfolio such that its liability positions are more exposed to asset revaluations. The government optimally borrows such that it is more indebted in good periods and less so in bad times as long-term debt's market value falls when government spending is high. This reduction in long-term prices serves as additional government financing.

Tax rates are kept perfectly smooth at 17.9%. Consumption is 9% higher while GDP (labor supply) is 11% lower in low-spending states relative to high spending states. Average GDP is $.95 \times 1.04 + .05 \times 1.15 = 1.0455$. The consolidated government's maturity structure consists of short-term assets (household borrowing) amounting to 384% of average GDP and long-term debt (household lending) amounting to 750% of average GDP. This consolidated maturity structure remains constant throughout time even though the central bank's and debt-manager's choices vary with changes in government spending. Remittances from the central bank to the debt-manager amount to 17% and 13% of average GDP in high-spending states depending on the previous state. They amount to amount to 0% and -2% of average GDP in low-spending states depending on the previous state. The central bank receives additional financing from the debt-manager along the Ramsey plan. Additional results are displayed in Table 4.

Variable	Description	Result
$(\tau(\ell)^*, \tau(h)^*)$	Tax rates	$(.179, .179)$
$(z(\ell)^*, z(h)^*)$	Ramsey transfers	$(0, -.665)$
$(c(\ell)^*, c(h)^*)$	Consumption	$(.868, .790)$
$(n(\ell)^*, n(h)^*)$	Labor supply	$(1.04, 1.15)$
$\boldsymbol{b}^* = \boldsymbol{b}(\ell)^* = \boldsymbol{b}(h)^*$	Outstanding maturity structure	$(-4.02, 7.84)$
$\hat{b}(\ell)^*$	DM issued debt in the low spending state	$(.244, .201)$
$\hat{b}(h)^*$	DM issued debt in the high spending state	$(-.260, .168)$
$\tilde{b}(\ell)^*$	CB held debt in the low spending state	$(4.26, -7.63)$
$\tilde{b}(h)^*$	CB held debt in the high spending state	$(3.76, -7.67)$
$\left(q^{(1)}(\ell), q^{(14)}(\ell), q^{(15)}(\ell)\right)$	Asset prices in the low spending state	$(.956, .512, .488)$
$\left(q^{(1)}(h), q^{(14)}(\ell), q^{(15)}(h)\right)$	Asset prices in the high spending state	$(.921, .428, .407)$
$(\Delta_{\ell,\ell}, \Delta_{\ell,h})$	Remittances in the low state after arriving from state s	$(.004, -.516)$
$(\Delta_{h,\ell}, \Delta_{h,h})$	Remittances in the high state after arriving from state s	$(.652, .135)$

Table 4: Results under the effectively complete markets specification

Short-term debt prices are 4% higher and long-term debt prices are 17% higher in low-spending states relative to their prices in high-spending states. Long-term bondholders enjoy a large return on their debt when the economy moves from a high marginal utility state to a low marginal utility one, but receive a negative return when the economy moves in the opposite direction. The gains and losses in households debt portfolios serve as lump-sum taxes and transfers and drive the choice of the welfare-maximizing central bank to steer the consolidated government to borrow in long-term debt and lend in short-term debt (hold short-term assets) as seen in Figure 2.

This follows a standard result in the literature: the covariance between real rates (inverse of debt prices) and government spending determines the optimal outstanding maturity structure, which is pointed out in Angeletos (2002), Bhandari et al. (2017) and others. When higher government spending is associated with lower debt prices (higher real interest rates), a short-term asset/long-term debt maturity structure is optimal (absent other sources of uncertainty or frictions).

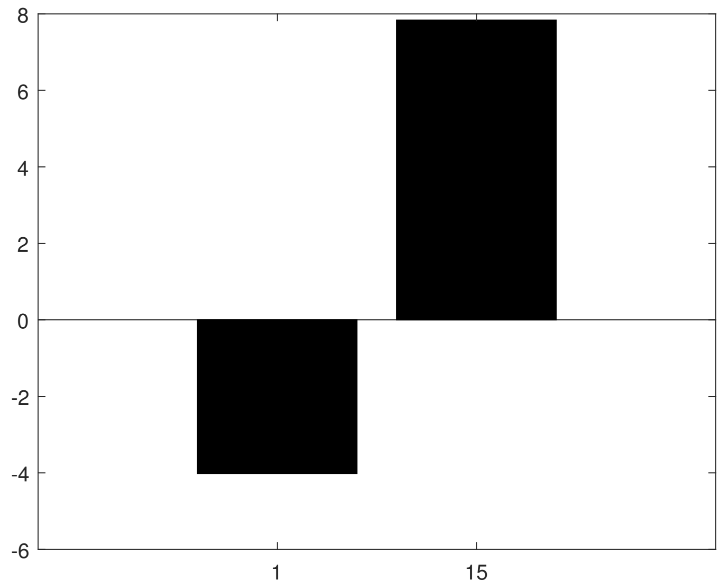
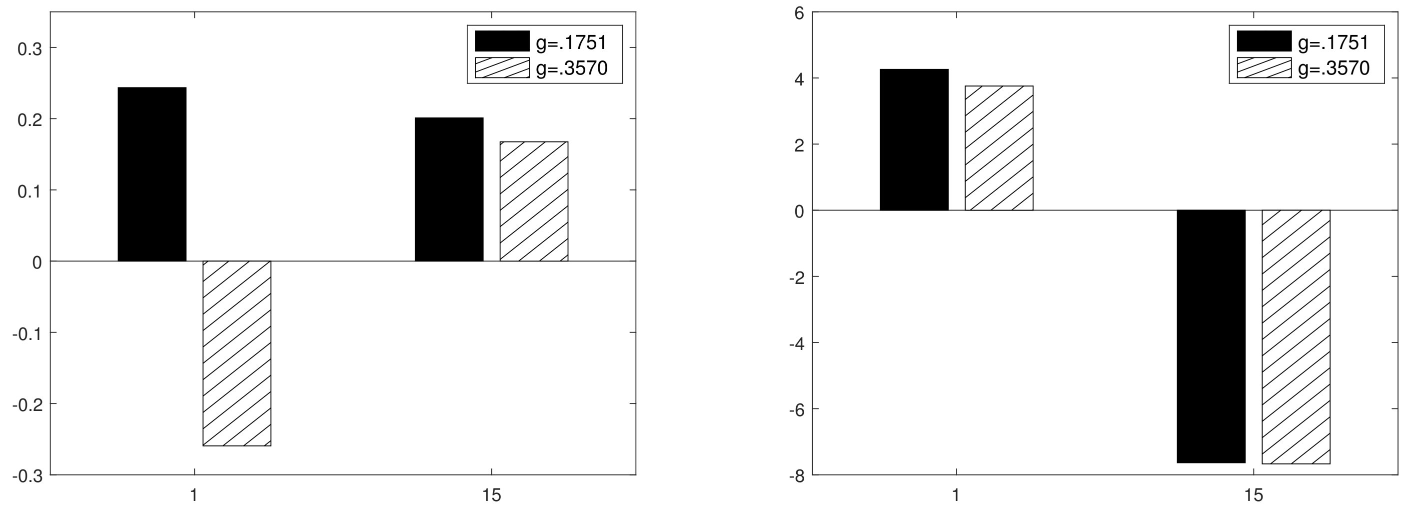


Figure 2: Outstanding Maturity Structure, $\boldsymbol{b}^*$

The strategic interaction between the two institutions does not result in a deviation from the planner's allocation, which is consistent with Proposition 1. The debt-manager borrows in long-term debt in all states, while also borrowing in short-term debt in low-spending states and lending in short-term debt in high-spending states. Regardless of state, the central bank issues large amounts of its own long-term debt, adding to the debt-manager's issuance. It uses a fraction of these borrowed resources to lend to the representative agent in short-term debt. This is the opposite of what modern-day central banks do on the open market. In the data central banks hold long-term government debt while large amounts of reserves (short-term debt) are held by households.

When the central bank holds large amounts of short-term debt, the consolidated government moves to a short position in short-term debt. The institutions jointly achieve the required outstanding structure $\hat{b} - \tilde{b} = \boldsymbol{b}^*$. Figure 3 displays institutional maturity structures. Positive bars in the left panel represent debt issuance and positive bars in the right panel represent central bank holdings (debt redemption, or debt netted out from the debt-manager's issuance).



(a) Debt-Manager Issued Debt (b) Central Bank Held Debt

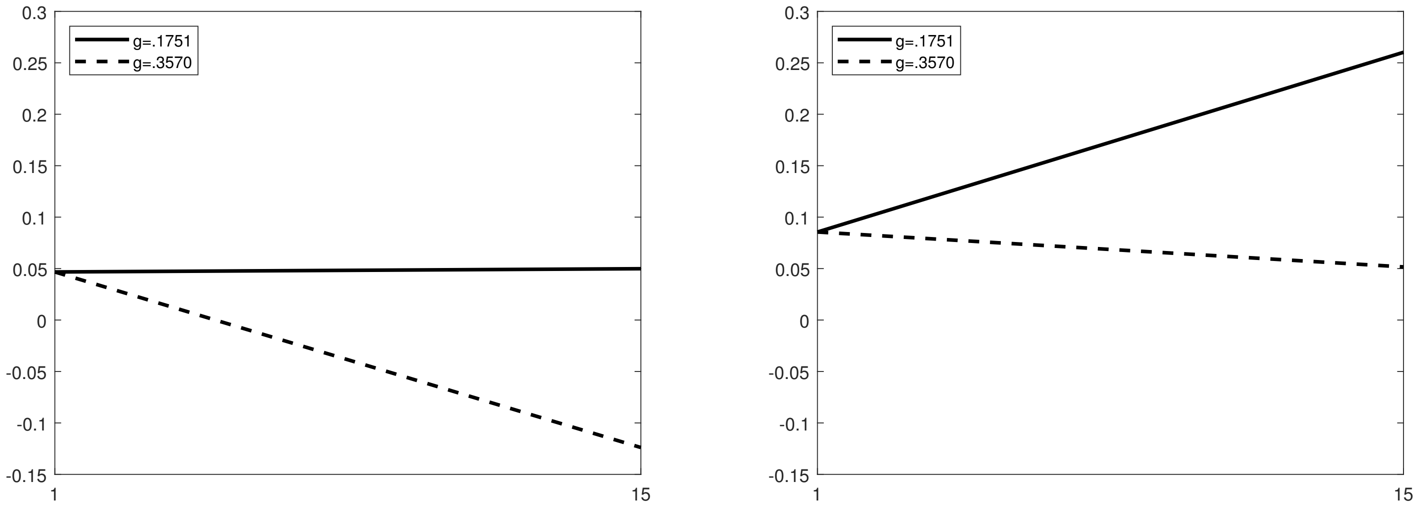
Figure 3: Institutional Maturity Structures

The Ramsey solution suggests a perfectly smoothed tax rate across all time periods which is achieved by the consolidated government. Households optimize both their consumption demand and labor supply decisions against the contemporaneous tax rate. With a higher tax rate τ_t comes a larger wedge between $u_1(c_t, n_t)$ and $u_2'(c_t, n_t)$. This is the lone distortion in the model. Because spending is lump-sum, the correlation between spending and tax's distortive properties is zero - the tax rate distorts the economy the same way regardless of what state is realized. Therefore the planner sets marginal welfare distortions equal across all states while satisfying the implementability constraint with an invariant tax rate.

Transitions in government spending drive asset revaluations through the aggregate resource constraint. Corresponding changes in marginal utilities (in both consumption and labor) flow through to asset prices. Economic states with higher government spending realize lower consumption and higher labor supply in equilibrium (government spending crowds out private consumption), so households optimally demand debt less in these periods. Figure 4 displays potential ex-post yield curves associated with being in a given state, outlining how these dynamics affect returns. The potential ex-post yield curves are calculated according to (17) using prices from the same debt object over two periods:

$$r_{t+1}^{(t+1)} = \frac{1 - q_t^{(t+1)}}{q_t^{(t+1)}} \quad , \quad r_{t+1}^{(t+15)} = \frac{q_{t+1}^{(t+15)} - q_t^{(t+15)}}{q_t^{(t+15)}}$$

Notice that the maturities of the short- and long-term bonds being redeemed in period $t+1$ are 0- and 14-periods respectively. Because the 1- and 15-period bonds have aged a period, they are valued according to their remaining life by optimizing households.



(a) Period-Ahead Yield Curve Possibilities when $g = .1751$

(b) Period-Ahead Yield Curve Possibilities when $g = .3570$

Figure 4: Ex-Post Yield Curves

Across all starting states as well as maturities, higher returns on debt are realized when government spending falls. Holding long-term assets in the low-spending regime results in negative returns upon reaching the high-spending one. The intuition for these results can be traced to the same intuition as above: households are both consuming less and working more when the government spends more, so saving is devalued and asset prices fall. Returns on short-term debt are guaranteed upon purchase because debt is non-contingent and indexed. There is no way for the government to devalue this type of debt because the price is guaranteed to be 1 in the next period.

7.2 Incomplete Markets Results

7.2.1 Debt-Manager Targets and Complete Market Preservation

As shown in Figure 3, the unconstrained welfare-maximizing central bank issues long-term debt in both states under the current calibration. When the short-selling constraints are added to the model, these issuances are bound below by zero. This constrains the welfare-maximizing institution making markets incomplete.

By definition of the Ramsey planner's solution, the incomplete markets outcome will always be worse than that from the complete market specification in terms of welfare. Before solving the model under incomplete markets, one may want to think about scenarios for which the short-selling constraints don't bind. One such way would be to explore the the debt-manager targets for which the debt-manager issues enough long-term debt so that the central bank holds long-term debt rather than issues it. Is this reasonable? The answer lies in Figure 5.

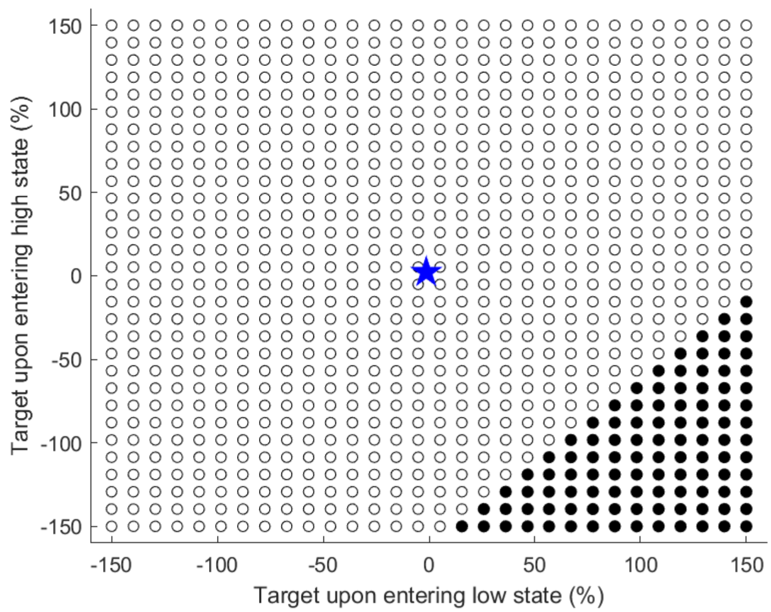


Figure 5: Debt-Manager Targets and Complete Markets

Empty circles indicate target combinations that result in moving to incomplete markets after adding the short-selling constraints. Solid dots represent targets for which these constraints do not bind. The calibrated debt service targets $K(s) = [k_\ell(s), k_h(s)] = [.02141, -.013536]$ are represented by the star towards the center of the figure.

The debt-manager would need to target massive interest outflows on its issued debt in low-spending states and massive interest inflows on its issued debt in high-spending states to negate binding short-selling constraints. These target combinations result in the debt-manager keeping a sufficient amount of long-term debt outstanding, which more closely matches the optimal consolidated allocation $\boldsymbol{b}^*$.

While it is possible for the debt-manager to target interest payments so as to keep the welfare-maximizing central bank from being bound by the short-selling constraints, interest inflows and outflows of 100% of the debt value is unreasonable.

7.2.2 Incomplete Markets Simulation Results

Normally when governments face an incomplete markets problem in this class of maturity management models, the reason the asset space doesn't span the state space is because the number of available assets is less than the number of possible exogenous states. This is the case in Barro (1979), Aiyagari et al. (2002), Bhandari et al. (2017) and many others. A small number of assets are asked to do the best they can at insuring the government against randomness. The result is random walk behavior in debt supply.

Market incompleteness does not come from the number of assets in the economy being less than the number of states in this analysis. Instead it comes from the government being unable to achieve the Ramsey allocation *despite* the fact that it has a sufficient number of assets at its disposal. This distinction is important because many variables in the model continue to be close-to-stable - a far cry from the results explored elsewhere in the literature, including the papers above.

Below is a set of time series from a model simulation where the central bank is constrained according to equation (23). The algorithm in section 6.3 is used to solve for these data. Sequences of outstanding debt $\{\boldsymbol{b_t}\}$, debt-manager issued debt $\{\hat{b}_t\}$, central bank held debt $\{\tilde{b}_t\}$ and remittance payments from the central bank to the debt-manager $\{\Delta_t\}$ are included. The final display is a comparison of welfare between the simulated economy under effectively complete markets, the incomplete markets via the short-selling constraint and financial autarky.

The simulation is 5000 periods long with a 500 period burn-in window. The average (among equations in system (27)) approximation error threshold for every period is .001. The mean of average errors over the simulation (after the burn-in period) is 2.08^{-07}.

Outstanding debt comes to rest around $\boldsymbol{b}_t = (-.07, -.02)$ when the economy is in a low-spending state for some time. The consolidated government holds a small asset position in both short- and long-term debt. Both outstanding short- and long-term debt sharply rise when the government switches to a high-spending regime. The government borrows in both debt objects initially to finance spending. Short-term debt continues to rise but long-term debt begins to fall if the economy stays in the high-spending regime for some time.

Short-term debt falls gradually until it converges back to its stable level of $-.07$ once the economy reverts to the low-spending regime. By contrast, long-term debt sharply falls after the regime change and quickly arrives at a large asset position well below its stable level of $-.02$. It converges back to this point from below. Figure 6 displays how outstanding debt evolves over the simulation. Shaded regions indicate high government spending states.

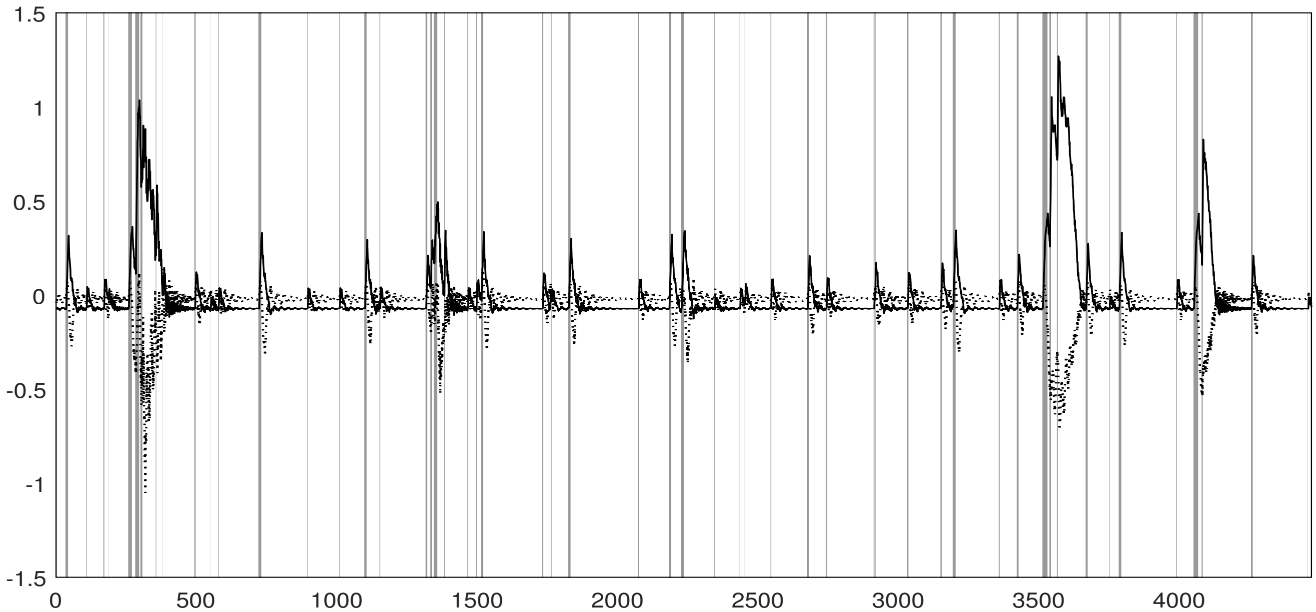


Figure 6: Short-Term (solid) and Long-Term (dotted) Debt Outstanding, $\boldsymbol{b}_t$

The debt-manager's issuance strategy continues to follow (19). Future debt prices conditional on regime are available and used to calculate the debt-manager's optimal choice to target ex-post returns in the following period thanks to the estimation algorithm's use of the forward states approach. Debt-manager issuance settles down at about $\hat{b}_t = (.27, .19)$ during times of low spending. So the issued maturity structure involves moderate, positive liability positions in both short- and long-term debt for the central government. When the economy enters a high-spending regime, the debt-manager shifts its issuance strategy to target a maturity structure that looks more like $\hat{b}_t = (-.21, .11)$. A total reversal of the way it looked when spending was low.

It issues less debt in both objects and even begins to save in short-term debt. Like in the complete markets case, both short- and long-term debt prices fall when the economy enters the high-spending state and rises again when the regime switches back. As such, the debt-manager's issuance allocations are similar in the two cases: moderate positive issuance in both debt types during good times and debt redemption (with short-term lending instead of borrowing) when debt becomes cheap. Figure 7 lays out the debt-manager's choices throughout the entire simulation. The longer high-spending states last, the more volatile debt prices become when the economy returns to normalcy. This affects the volatility of the optimizing, targeting debt-manager's choices.

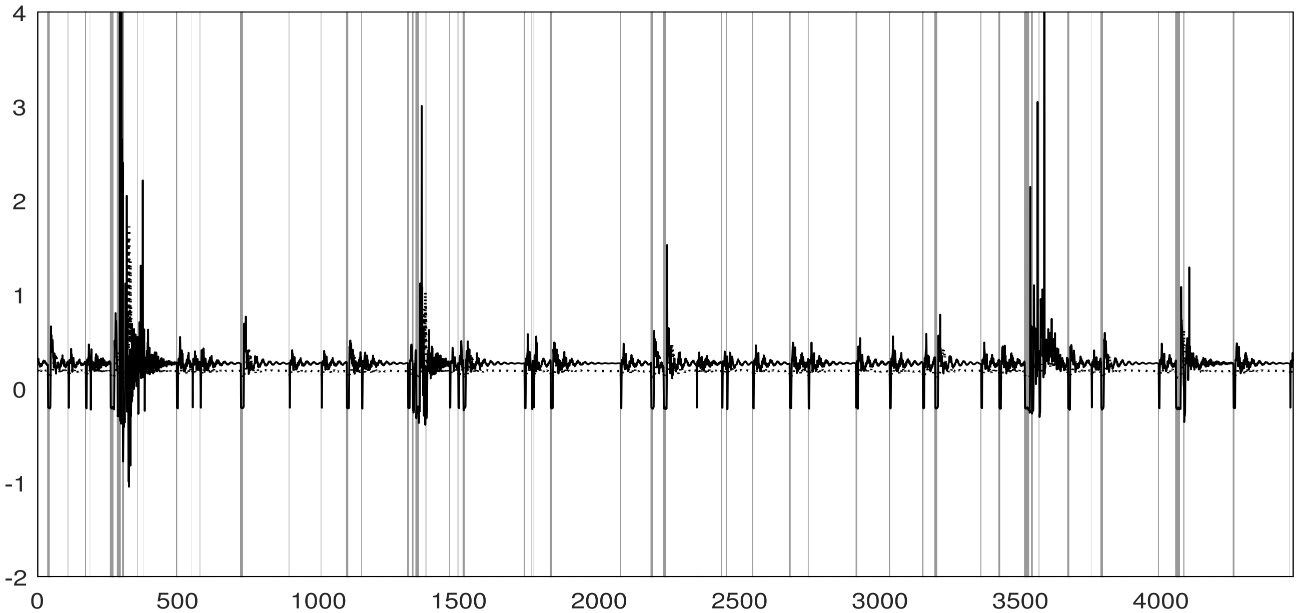


Figure 7: Short-Term (solid) and Long-Term (dotted) Debt-Manager Issued Debt, $\hat{b}_t$

The central bank manages its balance sheet to smooth taxes as much as it can given its short-selling constraint and its opponent's (the debt-manager's) choices. It understands that the welfare-maximizing outstanding maturity structure needs to follow that shown in Figure 6.

The consolidated government converges to a stable point when the economy stays in the low-spending regime for some time. That point is holding $\tilde{b}_t = (.34, .21)$ on its balance sheet and remitting interest profits back to the central government. This choice of debt-holdings more than negates the debt-manager's issuance resulting in the consolidated government's short- and long-term asset positions. Central bank decisions look a lot like those we see in the data when the economy enters the high-spending regime. Large influxes of short-term liabilities - quantitative easing. In windows where the economy sees many successive high-spending states, the central bank keeps issuing reserves while buying up long-term debt. This can be seen from periods 300-500 and from periods 3500-3600 in Figure 8.

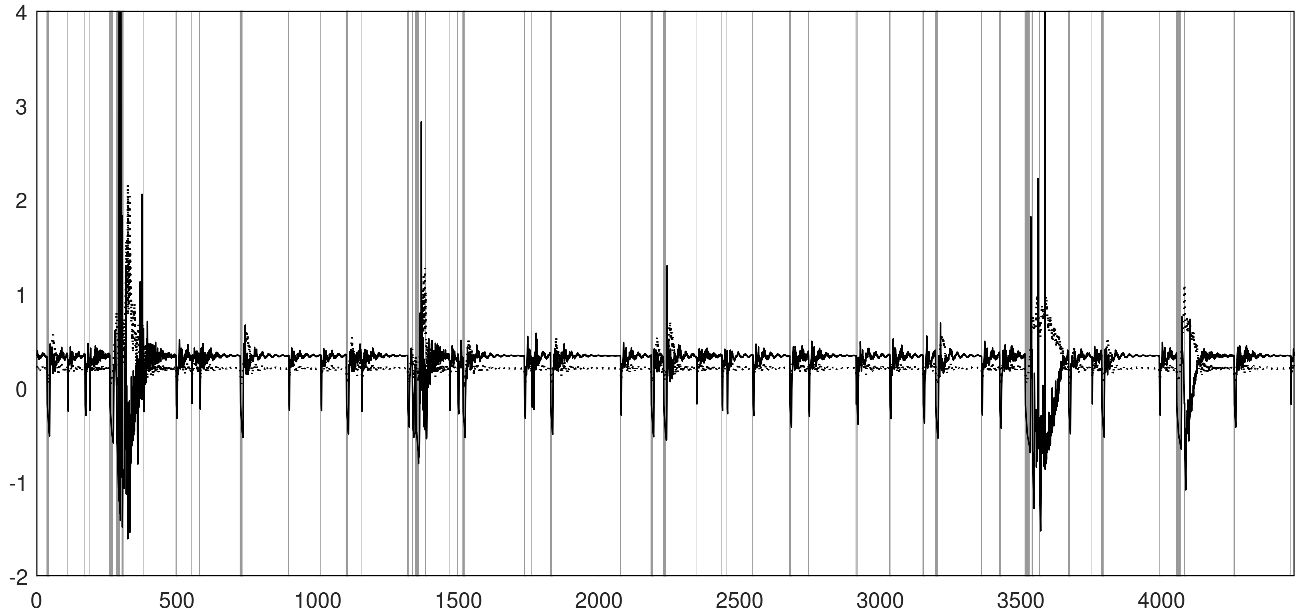


Figure 8: Short-Term (solid) and Long-Term (dotted) Central Bank Holdings, $\tilde{b}_t$

Remittances in this model remain frictionless when moving from effectively complete markets to incomplete markets. When the central bank decides to help fund the central government, it strategically buys less net debt than funds available to it (maturing debt). The difference is remitted to the government. When it wants to extract funds from the debt-manager to influence the tax rate, it does so by buying more debt than it has funds. Figure 9 displays how remittances from the central bank to the debt-manager progress over the simulation.

During a long period of low spending, remittances are a merely $\Delta_t = .02$. The central bank remits a small positive amount of funds to the debt-manager. When the economy enters a high-spending regime, remittances quickly and temporarily spike due to the large reserve issuance and debt sales the central bank executes. Remittances return close to zero even during the high spending state. Once the economy switches back to a low spending regime, remittances become highly and temporarily negative. The central bank uses these negative remittances to de-lever the consolidated government as spending is low again. If remittances were not allowed to be negative, this post-recovery transfer from the debt-manager to the central bank would not be possible and recovery would be tougher on households.

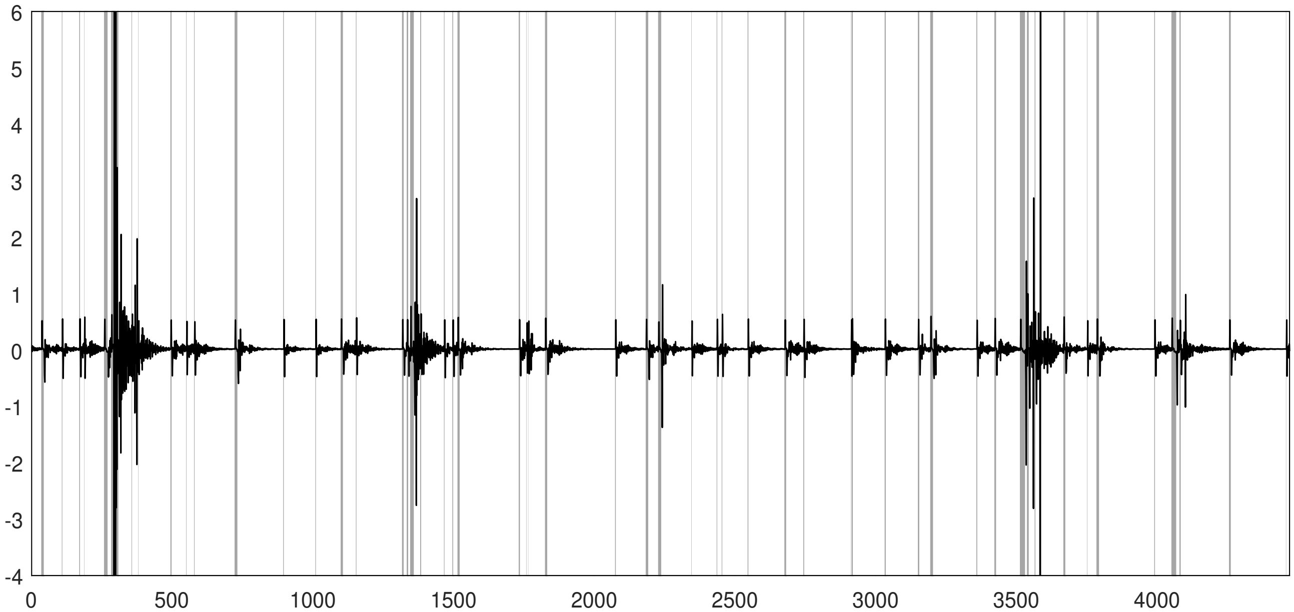


Figure 9: Remittances from the Central Bank to the Debt-Manager, Δ_t

In order to showcase how model objects return to their stable points after returning to the low-spending regime, Figure 10 displays periods 1830-2080 from the simulation. This includes the largest window for which the economy stayed in a low-spending regime allowing the model to return to its stable point.

Total debt outstanding has short-term debt converging from above after exiting a high-spending regime while long-term debt converges from below. The debt-manager goes from issuing a maturity structure with more long-term then short-term debt in the high-spending state to issuing its usual maturity structure. The central bank does the same with a sharp decrease in reserves immediately after the regime shift and a more gradual increase in long-term debt holdings. Eventually long-term debt falls back to its normal level. Remittances sharply fall after re-entering the low spending regime but quickly recover before converging back to .02.

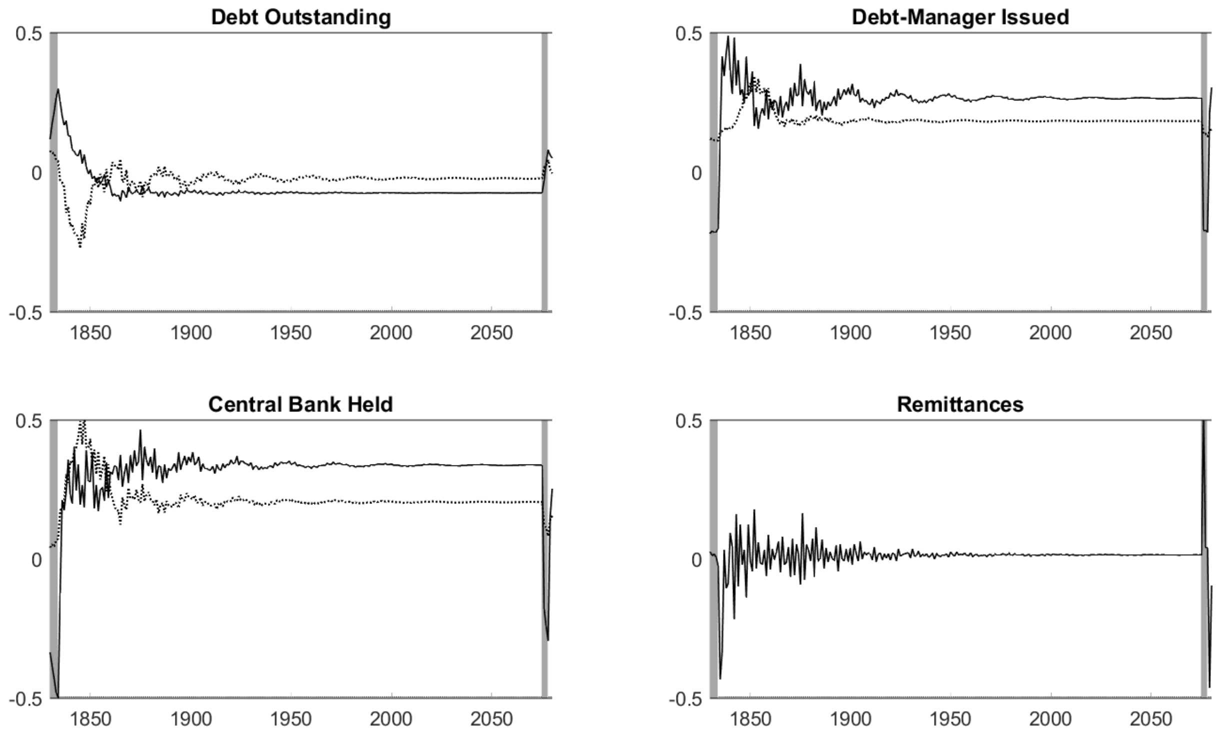


Figure 10: Panels on Model Stability. Short-term debt is solid, long-term debt is dotted in the first three panels.

In order to showcase how model objects react during a period of volatile regime shifts, Figure 10 showcases periods 3500-3580 from the simulation. This includes six regime switches.

Debt outstanding sees large and sharp increases in short-term debt + reserves during the bad states while long-term debt gradually falls. The debt-manager issues about the same level of long-term debt throughout the period while short-term debt sees strong variation with the regime switches. Central bank reserve issuance is quite large during each of the three high-spending regimes while long-term debt is being accumulated. Remittances follows its general pattern - large upward spikes upon entering the bad state and large downward spikes (going below zero) upon entering the good state.

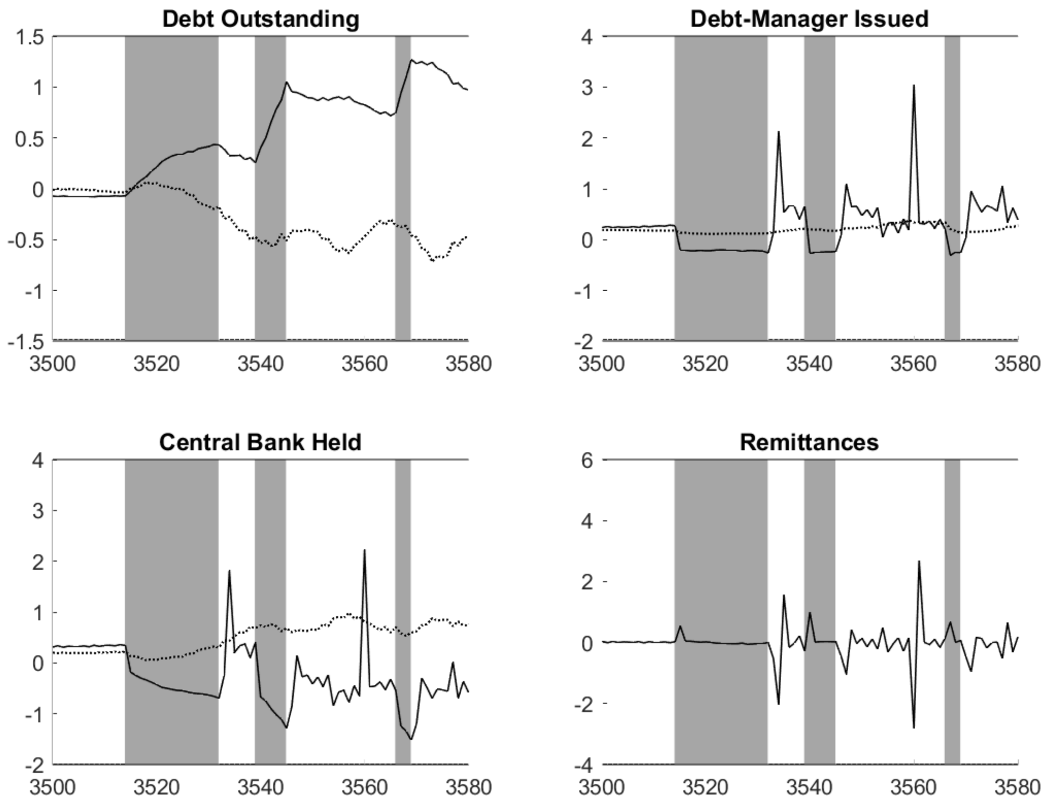


Figure 11: Panels on Debt Management During Bad States. Short-term debt is solid, long-term debt is dotted in the first three panels.

One main point of the paper is to explore the welfare cost of a single institutional constraint - the short-selling constraint on central bank-held long-term debt. This is a constraint that the Federal Reserve of the United States faces, but one that has been largely absent from modern macroeconomic analysis. The model developed in this paper is uniquely constructed to answer this question. Figure 12 shows the comparison. The utility households receive in the constrained simulation is closer to what it would be under total autarky than what it is under effectively complete markets. Households lose about 55% of the gains from complete debt markets in this model due to this institutional constraint. Utility in the low-spending states is highest in autarky, second highest in the incomplete markets setup and lowest under effectively complete markets. The value of the Ramsey solution lies entirely in the low utility states. Bad states are not as bad when the economy perfectly smooths taxes.

Standard macro theory with a portfolio return-targeting debt-manager and a welfare-maximizing central bank suggests that revoking the right for central banks to issue their own (or their county's) debt matters quite a bit. This is without implementing any sort of constraint on negative central bank remittances (which is a favorite among central governments).

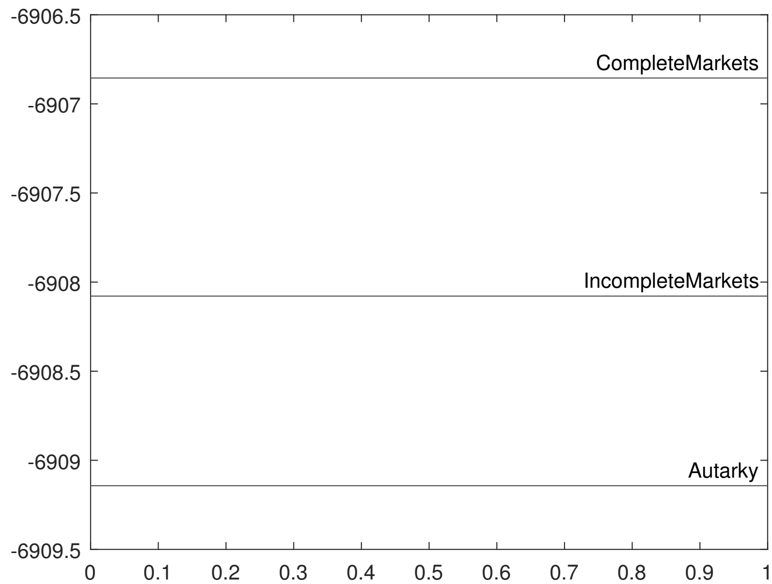


Figure 12: A Welfare Comparison

General patterns in optimal debt issuance arise in this simulation. When the central bank is welfare-maximizing but constrained to hold only positive levels of long-term government debt, the Ramsey planner's solution of perfect tax smoothing is unattainable. In order to get close, the central bank pushes the consolidated government to deal with spending shocks by increasing short-term debt + reserves while holding long-term assets. When spending is low, the central bank pushes the government to hold a small amount of both short- and long-term *assets*. Generally the optimal outstanding maturity structure is thought to always have more short-term debt (remittances + short-term government debt) than long-term debt. When spending is low, the theory suggests this is not the case.

Like in most all optimal debt management papers with a maturity structure, the way prices affect the value of outstanding debt plays the lead role. When spending rises, real rates increase and debt prices decrease. This is at the heart of why more long-term debt is outstanding during low-spending regimes and why more short-term debt is outstanding during high-spending regimes. Long-term debt valuation highly affects government financing and, thus, the distortionary tax rate.

8 Conclusion

In his 1932 statement to Congress, Irving Fisher proposed that "The [Federal Reserve] system is authorized to issue and sell...new interest-bearing debentures in such volume and of such date of maturity and rate of interest as may be deemed by it most suitable." This proposition was not accepted and has since fallen by the wayside of intellectual consideration in America. While the Fed can already issue ultra short-term debt in the form of bank reserves and currency, there's a major advantage in giving it the authority to issue long-term debt.[27]

Government indebtedness is measured by the market value of its outstanding and maturing debt. National financing changes with the level of indebtedness (whether from debt issuance/redemption or price movements of unmatured debt). Countries can strategically distribute sovereign debt across the maturity structure so that, given price movements, debt's market value endogenously evolves in a way advantageous to the taxpayer.[28] In a country like the US, this strategic management requires coordination between its federal economic institutions, especially when these institutions are constrained.[29]

A debt-manager and a central bank exist in the paper's model. These institutions are not independent, but their objectives are, which leads to the very coordination problem that can take a country away from managing its debt portfolio optimally. Model institutions are fully aware of how their opponent's actions affect things like central bank remittances, future debt prices and tax rates. Real-world institutions need to understand these issues as well if they want any hope of jointly determining the country's debt structure in a constructive way.

This analysis builds a model where the Ramsey planner's solution can be achieved even without institutional coordination, so long as a welfare-maximizing institution is unconstrained in its issuance and redemption of debt. The economy deviates from the first-best solution due to the incompleteness of financial markets when the welfare-maximizing institution becomes constrained. The type of market incompleteness seen in this model is importantly different from the types seen in a vast

[27] Whether this comes specifically in the form of US Treasuries or a different form of debt (call them Feddies), the backing of the debt would still be the American taxpayer and would thus be priced similarly.

[28] An example: When debt prices fall, they fall more for longer-term maturities. A decrease in price is a decrease in the market value of government debt and a reduction of government indebtedness. If managed correctly, price movements in the debt portfolio can be used as a source of financing for the government.

[29] In the model, the constraint on the central bank is obvious. The debt-manager is constrained by its non-welfare maximizing objective - much like many of the politically-led debt-managers in the world.

majority of maturity structure management papers. This incompleteness arises from an issuance sign restriction rather than from a lack of sufficient assets in the market. The implication is that the model retains large degrees of stability (stationarity) not seen in other incomplete market models.

Additionally, this paper showcases optimal quantitative easing in bad states of the world without any fancy bells or whistles needed. In a calibrated model of institutional non-cooperation in debt issuance/redemption and open market operations, the constrained welfare-maximizing central bank finds it necessary to issue large amounts of reserves over the course of high marginal utility regimes. Central bank long-term debt holdings increase as a result.

Finally, this analysis finds the true welfare cost of the short-selling constraint on a welfare-maximizing central bank. The simulated economy loses 55% of the utility difference between the Ramsey solution and total financial autarky when the central bank is not allowed to issue its own long-term debt. Dr. Fisher's proposal continues to be good policy a hundred years later.

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A Appendix: Proofs

Call the expected discounted lifelong utility of the representative household in equilibrium at time t 'social welfare' W_t. W_t is affected by debt strategies of government institutions and can be written as the function $W_t = W\left(\hat{b}_t, \tilde{b}_t | b_{t-1}, g(s)\right)$. Further, from the household's problem, debt market-clearing and the definition of strategies, $W\left(\hat{b}_t, \tilde{b}_t | b_{t-1}, g(s)\right)$ can be redefined as a function of the difference of the two strategy vectors $W\left(\hat{b}_t - \tilde{b}_t | b_{t-1}, g(s)\right) = W\left(\boldsymbol{b}_t | b_{t-1}, g(s)\right)$.

Let $\alpha_t \in \mathbb{R}^J$ be the unique W-maximizing maturity structure of debt at time t, subject to the consolidated government's budget constraint. So that $W(\alpha_t) > W(\bar{\alpha}_t)$, $\forall \bar{\alpha}_t$ which satisfy the consolidated GBC and where $\bar{\alpha}_t \neq \alpha_t$.

Proof of Lemma 1

Given information about an arbitrary $\tilde{b}_t$, Δ_t becomes solely a function of $\hat{b}_t$ such that the debt-manager is constrained by the consolidated GBC. Given the fact that the debt-manager's strategy space $\Gamma_{dm} = \mathbb{R}^I$, a welfare-maximizing debt-manager chooses its best response: $\hat{b}_t = \alpha_t + \tilde{b}_t$ where $\alpha_t \in \mathbb{R}^I$ and $\tilde{b}_t \in \mathbb{R}^I$. Then, the realized welfare function becomes $W_t = W\left(\alpha_t + \tilde{b}_t - \tilde{b}_t\right) = W(\alpha_t)$, which is the W-maximizing allocation subject to the consolidated GBC.

Proof of Lemma 2

Given information about an arbitrary $\hat{b}_t$, Δ_t becomes solely a function of $\tilde{b}_t$ such that the debt-manager is constrained by the consolidated GBC. Given the fact that the central bank's strategy space $\Gamma_{cb} = \mathbb{R}^I$, a welfare-maximizing debt-manager chooses its best response: $\tilde{b}_t = \alpha_t - \hat{b}_t \in \mathbb{R}^I$. The realized welfare function becomes $W_t = W\left(\hat{b}_t - \hat{b}_t - (-\alpha)\right) = W(\alpha_t)$, which is the W-maximizing allocation subject to the consolidated GBC.

Proof of Proposition 1

From Lemma 1 and Lemma 2, equilibrium debt allocations in an environment with a welfare-maximizing debt-manager or welfare-maximizing central bank only produce equilibria where $\hat{b}_t - \tilde{b}_t = \boldsymbol{b}_t = \alpha_t$.

Given the fact that the consolidated government's choice space is $\mathbb{R}^I$, a welfare-maximizing consolidated government (which is constrained only by the consolidated GBC by definition) chooses:

$\boldsymbol{b}_t = \alpha_t$ where $\alpha_t \in \mathbb{R}^I$. The realized welfare function becomes $W_t = W(\alpha_t)$, which is the W-maximizing allocation subject to the consolidated GBC.

Proof of Corollary 1

Let a welfare-maximizing central bank's strategy be initially fixed at an arbitrary vector $\tilde{b}_t^0 \in \mathbb{R}^I$ so that it faces a welfare-maximizing debt-manager's strategy $\hat{b}_t = \alpha_t + \tilde{b}_t^0$. From Lemma 2, the welfare-maximizing central bank's best response is $\tilde{b}_t = \hat{b}_t - \alpha_t = \alpha_t + \tilde{b}_t^0 - \alpha_t = \tilde{b}_t^0$. Therefore, once the welfare-maximizing central bank is free to choose its vector, neither the central bank nor debt-manager deviate from their initial strategies, constituting an equilibrium.

Proof of Corollary 2

Let a welfare-maximizing debt-manager's strategy be initially fixed at an arbitrary vector $\hat{b}_t^0 \in \mathbb{R}^I$ so that it faces a central bank's strategy $\tilde{b}_t = \hat{b}_t^0 - \alpha_t$. From Lemma 1, the welfare-maximizing debt-manager's best response is $\hat{b}_t = \alpha_t + \tilde{b}_t = \alpha_t + \hat{b}_t^0 - \alpha_t = \hat{b}_t^0$. Therefore, once the welfare-maximizing debt-manager is free to choose its vector, neither the central bank nor debt-manager deviate from their initial strategies, constituting an equilibrium.

Proof of Lemma 3

Repeat the proof from Corollary 1 J times, each time adding $\varepsilon \in \mathbb{R}$ to only one element of $\tilde{b}_t^0$. The proof is complete after doing the same for Corollary 2 and $\hat{b}_t^0$.

Proof of Proposition 2

First, define an $S \times 1$ vector of state-contingent government transfers to households given an arbitrary feasible tax plan as Z. Given a tax plan and a maturity structure of I maturities, define A to be the $S \times I$ payout matrix for the given feasible tax plan. When $I = S$, A is non-singular. Similar to (16), the identity for government transfers through a supporting debt position can be written as $Z = A\boldsymbol{b}_t$. Where $\boldsymbol{b}_t = \boldsymbol{b} = A^{-1}Z$ is a unique, time-invariant maturity structure that supports the feasible tax plan.

B Appendix: Solving the Ramsey Problem

Roadmap:

1. Solve for the household FOCs and express the tax system and price system in terms of allocations.

2. Use these FOCs to express the government budget constraint as a constraint on allocations.

3. Find the FOCs of the Ramsey planner to characterize the feasible allocation that maximizes welfare subject to the restriction from step 2.

4. Use the household FOCs from step 1 along with the equilibrium allocations from step 3 to solve for the equilibrium Ramsey plan.

Step 1 The household's FOCs yield:

$$\Lambda_t = \beta^t \frac{c_t^{-\sigma}}{c_0^{-\sigma}} \tag{28}$$

$$1 - \tau_t = \frac{n_t^{\varphi}}{c_t^{-\sigma}} \tag{29}$$

So that the take-home rate is equivalent to the ratio of marginal utility in labor to that in consumption.

Step 2 Using (28) and (29), rewrite (10) as:

$$0 = \mathbb{E} \sum_t^{\infty} \beta^t \left[(n_t - g_t)\, c_t^{-\sigma} - n_t^{1+\varphi} \right] \tag{30}$$

Which I call the planner's implementability constraint. This constraint outlines the government's discounted infinite stream of primary surpluses in terms of allocations and the stochastic process. The intuition is the same as above.

Step 3 Where implementability (30) and feasibility (7) are the relevant constraints, the planner's resulting Lagrangian is:

$$\mathcal{L} = \mathbb{E} \sum_t^{\infty} \beta^t \left\{ \frac{c_t^{1-\sigma}}{1-\sigma} - \frac{n_t^{1+\varphi}}{1+\varphi} \right.$$

$$+\lambda_0 \left[(n_t - g_t)\, c_t^{-\sigma} - n_t^{1+\varphi} \right]$$

$$+\mu_{0t} \left[n_t - c_t - g_t \right] \}$$

Taking FOCs of this Lagrangian yields:

$$c_t^{-\sigma} - \lambda_0 \left[\sigma c_t^{-1-\sigma} (n_t - g_t) \right] = \mu_{0t} \tag{21a}$$

$$n_t^{\varphi} + \lambda_0 \left[(1+\varphi)\, n_t^{\varphi} - c_t^{-\sigma} \right] = \mu_{0t} \tag{21b}$$

$$c_t = n_t - g_t \tag{21c}$$

Combining (21a) and (21b) through μ_{0t} results in:

$$c_t^{-\sigma} - \lambda_0 \left[\sigma c_t^{-1-\sigma} (n_t - g_t) \right] = n_t^{\varphi} + \lambda_0 \left[(1+\varphi)\, n_t^{\varphi} - c_t^{-\sigma} \right]$$

And simplifying yields:

$$n_t^{\varphi} = \left[\frac{1 + \lambda_0 (1-\sigma)}{1 + \lambda_0 (1+\varphi)} \right] c_t^{-\sigma}$$

For simplicity, define the constant $\tilde{\lambda} = \left[\frac{1 + \lambda_0 (1-\sigma)}{1 + \lambda_0 (1+\varphi)} \right]$ to write:

$$n_t^{\varphi} = \tilde{\lambda} c_t^{-\sigma} \tag{22}$$

Notice that (22) and (29) imply $1 - \tau_t = \tilde{\lambda}$. The tax rate will be held constant by the planner.

Using the feasability constraint (21c), (22) can be converted into a relation between the optimal c_t and g_t:

$$(c_t + g_t)^{\varphi} = \tilde{\lambda} c_t^{-\sigma} \tag{23}$$

Define $c_t^* \equiv c(\tilde{\lambda}, g_t)$ as the solution to (23).

Substitute this solution back into the implementability constraint (30) to solve for $\tilde{\lambda}$ as a function of the process for g_t:

$$0 = \mathbb{E} \sum_{t}^{\infty} \beta^t \left[(c_t^*)^{1-\sigma} - (c_t^* + g_t)^{1+\varphi} \right] \tag{24}$$

Step 4 Define $\tilde{\lambda}^*$ as the $\tilde{\lambda}$ that solves (24). The Ramsey consumption allocation c_t^* is then solved for using $\tilde{\lambda}^*$ and (23). And the Ramsey labor allocation is solved for using $\tilde{\lambda}^*$, c_t^* and (22). The Ramsey tax plan is solved for using (29).

C Appendix: Deriving Closed Solutions Under Complete Markets

In the two-state, two-maturity model, Z^* and A^* become:

$$Z^* = \begin{bmatrix} z(\ell)^* & z(h)^* \end{bmatrix}' \quad , \quad A^* = \begin{bmatrix} 1 & q_t^{(t+J-1)}(\ell)^* \\ 1 & q_t^{(t+J-1)}(h)^* \end{bmatrix}$$

So that:

$$Z^* = A^* \boldsymbol{b}_{t-1} = \begin{bmatrix} z(\ell)^* \\ z(h)^* \end{bmatrix} = \begin{bmatrix} 1 & q_t^{(t+J-1)}(\ell)^* \\ 1 & q_t^{(t+J-1)}(h)^* \end{bmatrix} \begin{bmatrix} \boldsymbol{b}_{t-1}^{(t)^*} \\ \boldsymbol{b}_{t-1}^{(t+1)^*} \end{bmatrix}$$

Premultiplying both sides of the equation by ${A^*}^{-1}$ yields the optimal outstanding maturity structure $\boldsymbol{b}^*$:

$$\begin{bmatrix} \boldsymbol{b}_{t-1}^{(t)^*} \\ \boldsymbol{b}_{t-1}^{(t+1)^*} \end{bmatrix} = \begin{bmatrix} 1 & q_t^{(t+J-1)}(\ell)^* \\ 1 & q_{t,h}^{(t+J-1)}(h)^* \end{bmatrix}^{-1} \begin{bmatrix} z(\ell)^* \\ z(h)^* \end{bmatrix} = \left(\frac{1}{q_t^{(t+J-1)}(h)^* - q_t^{(t+J-1)}(\ell)^*} \right) \begin{bmatrix} q_t^{(t+J-1)}(\ell)^* & -q_t^{(t+J-1)}(h)^* \\ -1 & 1 \end{bmatrix} \begin{bmatrix} z(\ell)^* \\ z(h)^* \end{bmatrix}$$

Iterating forward to time t, this can be written in closed-form as:

$$\boldsymbol{b}_t^{(t+1)^*} = \left(\frac{q_{t+1}^{(t+J)}(h)^*}{q_{t+1}^{(t+J)}(h)^* - q_{t+1}^{(t+J)}(\ell)^*} \right) z(\ell)^* - \left(\frac{q_{t+1}^{(t+J)}(\ell)^*}{q_{t+1}^{(t+J)}(h)^* - q_{t+1}^{(t+J)}(\ell)^*} \right) z(h)^*$$

$$\boldsymbol{b}_t^{(t+2)^*} = -\left(\frac{1}{q_{t+1}^{(t+J)}(h)^* - q_{t+1}^{(t+J)}(\ell)^*} \right) z(\ell)^* + \left(\frac{1}{q_{t+1}^{(t+J)}(h)^* - q_{t+1}^{(t+J)}(\ell)^*} \right) z(h)^*$$

$K(s)$ and $R\,(s)^*$ become:

$$K(s) = \begin{bmatrix} k_{t,\ell}(s) & k_{t,h}(s) \end{bmatrix}' \quad , \quad R\,(s)^* = \begin{bmatrix} \dfrac{1 - q_{t-1}^{(t)}\,(s)^*}{q_{t-1}^{(t)}\,(s)^*} & \dfrac{q_t^{(t+J-1)}\,(\ell)^* - q_{t-1}^{(t+J-1)}\,(s)^*}{q_{t-1}^{(t+J-1)}\,(s)^*} \\ \dfrac{1 - q_{t-1}^{(t)}\,(s)^*}{q_{t-1}^{(t)}\,(s)^*} & \dfrac{q_t^{(t+J-1)}\,(h)^* - q_{t-1}^{(t+J-1)}\,(s)^*}{q_{t-1}^{(t+J-1)}\,(s)^*} \end{bmatrix}$$

So that:

$$K(s) = R\,(s)^*\,\hat{b}\,(s)^* = \begin{bmatrix} k_{t,\ell}(s) \\ k_{t,h}(s) \end{bmatrix} = \begin{bmatrix} \dfrac{1 - q_{t-1}^{(t)}\,(s)^*}{q_{t-1}^{(t)}\,(s)^*} & \dfrac{q_t^{(t+J-1)}\,(\ell)^* - q_{t-1}^{(t+J-1)}\,(s)^*}{q_{t-1}^{(t+J-1)}\,(s)^*} \\ \dfrac{1 - q_{t-1}^{(t)}\,(s)^*}{q_{t-1}^{(t)}\,(s)^*} & \dfrac{q_t^{(t+J-1)}\,(h)^* - q_{t-1}^{(t+J-1)}\,(s)^*}{q_{t-1}^{(t+J-1)}\,(s)^*} \end{bmatrix} \begin{bmatrix} \hat{b}_{t-1}^{(t)}(s)^* \\ \hat{b}_{t-1}^{(t+J-1)}(s)^* \end{bmatrix}$$

Premultiplying both sides of the equation by $R^{*^{-1}}$ yields the optimal debt-manager choice of issuance $\hat{b}(s)^*$:

$$\begin{bmatrix} \hat{b}_{t-1}^{(t)}(s)^* \\ \hat{b}_{t-1}^{(t+J-1)}(s)^* \end{bmatrix} = \begin{bmatrix} \dfrac{1 - q_{t-1}^{(t)}\,(s)^*}{q_{t-1}^{(t)}\,(s)^*} & \dfrac{q_t^{(t+J-1)}\,(\ell)^* - q_{t-1}^{(t+J-1)}\,(s)^*}{q_{t-1}^{(t+J-1)}\,(s)^*} \\ \dfrac{1 - q_{t-1}^{(t)}\,(s)^*}{q_{t-1}^{(t)}\,(s)^*} & \dfrac{q_t^{(t+J-1)}\,(h)^* - q_{t-1}^{(t+J-1)}\,(s)^*}{q_{t-1}^{(t+J-1)}\,(s)^*} \end{bmatrix}^{-1} \begin{bmatrix} k_{t,\ell}(s) \\ k_{t,h}(s) \end{bmatrix}$$

$$= \left(\frac{1}{\dfrac{1 - q_{t-1}^{(t)}\,(s)^*}{q_{t-1}^{(t)}\,(s)^*} \left[\dfrac{q_t^{(t+J-1)}\,(h)^* - q_t^{(t+J-1)}\,(\ell)^*}{q_{t-1}^{(t+J-1)}\,(s)^*} \right]} \right) \begin{bmatrix} \dfrac{q_t^{(t+J-1)}\,(h)^* - q_{t-1}^{(t+J-1)}\,(s)^*}{q_{t-1}^{(t+J-1)}\,(s)^*} & -\dfrac{q_t^{(t+J-1)}\,(\ell)^* - q_{t-1}^{(t+J-1)}\,(s)^*}{q_{t-1}^{(t+J-1)}\,(s)^*} \\ -\dfrac{1 - q_{t-1}^{(t)}\,(s)^*}{q_{t-1}^{(t)}\,(s)^*} & \dfrac{1 - q_{t-1}^{(t)}\,(s)^*}{q_{t-1}^{(t)}\,(s)^*} \end{bmatrix} \begin{bmatrix} k_{t,\ell}(s) \\ k_{t,h}(s) \end{bmatrix}$$

Which can also be written as:

$$\hat{b}_{t-1}^{(t)}(s)^* = \left(\frac{q_{t-1}^{(t)}\,(s)^* \left(q_t^{(t+J-1)}\,(h)^* - q_{t-1}^{(t+J-1)}\,(s)^* \right)}{\left(1 - q_{t-1}^{(t)}\,(s)^*\right) \left(q_t^{(t+J-1)}\,(h)^* - q_t^{(t+J-1)}\,(\ell)^* \right)} \right) k_{t,\ell}(s) - \left(\frac{q_{t-1}^{(t)}\,(s)^* \left(q_t^{(t+J-1)}\,(\ell)^* - q_{t-1}^{(t+J-1)}\,(s)^* \right)}{\left(1 - q_{t-1}^{(t)}\,(s)^*\right) \left(q_t^{(t+J-1)}\,(h)^* - q_t^{(t+J-1)}\,(\ell)^* \right)} \right) k_{t,h}(s)$$

$$\hat{b}_{t-1}^{(t+J-1)}(s)^{*} = -\left(\frac{q_{t-1}^{(t+J-1)}(s)^{*}}{q_{t}^{(t+J-1)}(h)^{*} - q_{t}^{(t+J-1)}(\ell)^{*}}\right) k_{t,\ell}(s) + \left(\frac{q_{t-1}^{(t+J-1)}(s)^{*}}{q_{t}^{(t+J-1)}(h)^{*} - q_{t}^{(t+J-1)}(\ell)^{*}}\right) k_{t,h}(s)$$

Iterating t forward by one gives us:

$$\hat{b}_{t}^{(t+1)}(s)^{*} = \left(\frac{q_{t}^{(t+1)}(s)^{*}\left(q_{t+1}^{(t+J)}(h)^{*} - q_{t}^{(t+J)}(s)^{*}\right)}{\left(1 - q_{t}^{(t+1)}(s)^{*}\right)\left(q_{t+1}^{(t+J)}(h)^{*} - q_{t+1}^{(t+J)}(\ell)^{*}\right)}\right) k_{t+1,\ell}(s) - \left(\frac{q_{t}^{(t+1)}(s)^{*}\left(q_{t+1}^{(t+J)}(\ell)^{*} - q_{t}^{(t+J)}(s)^{*}\right)}{\left(1 - q_{t}^{(t+1)}(s)^{*}\right)\left(q_{t+1}^{(t+J)}(h)^{*} - q_{t+1}^{(t+J)}(\ell)^{*}\right)}\right) k_{t+1,h}(s)$$

$$\hat{b}_{t}^{(t+J)}(s)^{*} = -\left(\frac{q_{t}^{(t+J)}(s)^{*}}{q_{t+1}^{(t+J)}(h)^{*} - q_{t+1}^{(t+J)}(\ell)^{*}}\right) k_{t+1,\ell}(s) + \left(\frac{q_{t}^{(t+J)}(s)^{*}}{q_{t+1}^{(t+J)}(h)^{*} - q_{t+1}^{(t+J)}(\ell)^{*}}\right) k_{t+1,h}(s)$$

The central bank's best response function is simply:

$$\tilde{b}(s)^{*} = \hat{b}(s)^{*} - \boldsymbol{b}^{*}$$

Which can be written as:

$$\tilde{b}_{t}^{(t+1)}(s)^{*} = \left(\frac{q_{t}^{(t+1)}(s)^{*}\left(q_{t+1}^{(t+J)}(h)^{*} - q_{t}^{(t+J)}(s)^{*}\right)}{\left(1 - q_{t}^{(t+1)}(s)^{*}\right)\left(q_{t+1}^{(t+J)}(h)^{*} - q_{t+1}^{(t+J)}(\ell)^{*}\right)}\right) k_{t+1,\ell}(s) - \left(\frac{q_{t}^{(t+1)}(s)^{*}\left(q_{t+1}^{(t+J)}(\ell)^{*} - q_{t}^{(t+J)}(s)^{*}\right)}{\left(1 - q_{t}^{(t+1)}(s)^{*}\right)\left(q_{t+1}^{(t+J)}(h)^{*} - q_{t+1}^{(t+J)}(\ell)^{*}\right)}\right) k_{t+1,h}(s)$$

$$- \left(\frac{q_{t+1}^{(t+J)}(h)^{*}}{q_{t+1}^{(t+J)}(h)^{*} - q_{t+1}^{(t+J)}(\ell)^{*}}\right) z(\ell)^{*} + \left(\frac{q_{t+1}^{(t+J)}(\ell)^{*}}{q_{t+1}^{(t+J)}(h)^{*} - q_{t+1}^{(t+J)}(\ell)^{*}}\right) z(h)^{*}$$

$$\tilde{b}_{t}^{(t+2)}(s)^{*} = -\left(\frac{q_{t}^{(t+J)}(s)^{*}}{q_{t+1}^{(t+J)}(h)^{*} - q_{t+1}^{(t+J)}(\ell)^{*}}\right) k_{t+1,\ell}(s) + \left(\frac{q_{t}^{(t+J)}(s)^{*}}{q_{t+1}^{(t+J)}(h)^{*} - q_{t+1}^{(t+J)}(\ell)^{*}}\right) k_{t+1,h}(s)$$

$$+ \left(\frac{1}{q_{t+1}^{(t+J)}(h)^{*} - q_{t+1}^{(t+J)}(\ell)^{*}}\right) z(\ell)^{*} - \left(\frac{1}{q_{t+1}^{(t+J)}(h)^{*} - q_{t+1}^{(t+J)}(\ell)^{*}}\right) z(h)^{*}$$

D Appendix: Deriving the Measurability Constraint

To arrive at the government's measurability constraint, combine the household's FOCs in c_t, n_t and b_t, the aggregate resource constraint and the government budget constraint.

HHFOC (c_t):

$$\lambda_t = c_t^{-\sigma}$$

HHFOC (n_t):

$$n_t^{\varphi} = \lambda_t(1-\tau_t)$$

HHFOC (b_t):

$$q_t^{(t+j)} = \frac{\beta^j \mathbb{E}_t \lambda_{t+j}}{\lambda_t} \quad \forall\ j \in \{1\ ,\ \cdots\ ,\ J\}$$

ARC:

$$c_t + g_t = n_t$$

Consolidated GBC:

$$\sum_{i=0}^{J} q_{t+i} \boldsymbol{b}_t^{(t+i)} + \tau_t n_t - g_t = \sum_{i=0}^{J} q_{t+i} \boldsymbol{b}_{t-1}^{(t+i+1)}$$

Combining the first three equations gives:

$$\tau_t = 1 - \frac{n_t^{\varphi}}{c_t^{-\sigma}}$$

$$q_t^{(t+j)} = \frac{\beta^j \mathbb{E}_t \left[c_{t+j}^{-\sigma}\right]}{c_t^{-\sigma}} \quad \forall\ j \in \{1\ ,\ \cdots\ ,\ J\}$$

Using these two equations to eliminate τ_t and $q_t^{(t+j)}$ and using the ARC to eliminate n_t from the consolidated GBC yields the measurability constraint:

$$\sum_{j=1}^{J} \beta^j \mathbb{E}_t \left[c_{t+j}^{-\sigma}\right] \boldsymbol{b}_t^{(t+j)} - \sum_{j=1}^{J} \beta^{j-1} \mathbb{E}_t \left[c_{t+j-1}^{-\sigma}\right] \boldsymbol{b}_{t-1}^{(t+j-1)} = (c_t + g_t)^{1+\varphi} - c_t^{1-\sigma}$$

E Appendix: Outlining Incomplete Markets Estimation

E.1 Lagrange Multipliers and the State Space

The system (27) includes additional Lagrange multipliers from the institutional non-negativity constraint and the Milar bound constraints: $\zeta_{L,t}^{(j)}$, $\zeta_{U,t}^{(j)}$, $\zeta_{L,t}^{\text{Total}}$, $\zeta_{U,t}^{\text{Total}}$, $\mu_t^{(j)}$. In order to simplify the estimation, these multipliers are approximated according the functions:

$$\zeta_{L,t}^{(j)} = \phi\left(\underline{M}^{(j)} - \boldsymbol{b}_t^{(t+j)}\right) + \ln\left(1 + \phi\left(\underline{M}^{(j)} - \boldsymbol{b}_t^{(t+j)}\right)\right) \quad \text{when} \quad \boldsymbol{b}_t^{(t+j)} < \underline{M}^{(j)}$$

$$\zeta_{U,t}^{(j)} = \phi\left(\boldsymbol{b}_t^{(t+j)} - \overline{M}^{(j)}\right) + \ln\left(1 + \phi\left(\boldsymbol{b}_t^{(t+j)} - \overline{M}^{(j)}\right)\right) \quad \text{when} \quad \boldsymbol{b}_t^{(t+j)} > \overline{M}^{(j)}$$

$$\zeta_{L,t}^{\text{Total}} = \phi\left(\underline{M}^{\text{Total}} - \sum_{j=1}^{J}\boldsymbol{b}_t^{(t+j)}\right) + \ln\left(1 + \phi\left(\underline{M}^{\text{Total}} - \sum_{j=1}^{J}\boldsymbol{b}_t^{(t+j)}\right)\right) \quad \text{when} \quad \sum_{j=1}^{J}\boldsymbol{b}_t^{(t+j)} < \underline{M}^{\text{Total}}$$

$$\zeta_{U,t}^{\text{Total}} = \phi\left(\sum_{j=1}^{J}\boldsymbol{b}_t^{(t+j)} - \overline{M}^{\text{Total}}\right) + \ln\left(1 + \phi\left(\sum_{j=1}^{J}\boldsymbol{b}_t^{(t+j)} - \overline{M}^{\text{Total}}\right)\right) \quad \text{when} \quad \sum_{j=1}^{J}\boldsymbol{b}_t^{(t+j)} > \overline{M}^{\text{Total}}$$

$$\mu_t^{(j)} = \phi\left(\boldsymbol{b}_t^{(t+j)} - \hat{b}_t^{(t+j)}\right) + \ln\left(1 + \phi\left(\boldsymbol{b}_t^{(t+j)} - \hat{b}_t^{(t+j)}\right)\right) \quad \text{when} \quad \boldsymbol{b}_t^{(t+j)} > \hat{b}_t^{(t+j)} \quad \forall\, j \in \{j_2\,,\ \cdots\,,\ J\}$$

and equal to zero otherwise. $\phi = 90$ in the code.

As noted in Valaitis and Villa (2023), adding these approximated multipliers to the state space improves accuracy. As such, the state space becomes:

$$X_t = \left\{g_t, \{\lambda_{t-i}\}_{i=1}^{J}, \left\{\left\{\boldsymbol{b}_{t-i}^{(t)}\right\}_{i=1}^{j}\right\}_{j=1}^{J}, \left\{\left\{\zeta_{L,t-i}^{(j)}\right\}_{i=1}^{j}\right\}_{j=1}^{J}, \left\{\left\{\zeta_{U,t-i}^{(j)}\right\}_{i=1}^{j}\right\}_{j=1}^{J}, \zeta_{L,t-1}^{\text{Total}}, \zeta_{U,t-1}^{\text{Total}}, \left\{\mu_{t-1}^{(j)}\right\}_{j=j_2}^{J}\right\}$$

When $S = I = 2$ and $J = 15$, the state space includes 67 variables.

E.2 Forward States Approach

We would like to approximate the following 5 expectations:

$$\mathbb{E}_t\left[c_{t+1}^{-\sigma}\right] \quad , \quad \mathbb{E}_t\left[c_{t+1}^{-\sigma}\lambda_{t+1}\right] \quad , \quad \mathbb{E}_t\left[c_{t+J}^{-\sigma}\right] \quad , \quad \mathbb{E}_t\left[c_{t+J}^{-\sigma}\lambda_{t+J}\right] \quad , \quad \mathbb{E}_t\left[c_{t+J-1}^{-\sigma}\right]$$

using the state space of pre-determined variables X_t.

So that:

$$E_t\left[c_{t+1}^{-\sigma}\right] = \mathcal{N}\mathcal{N}_1^{(1)}\left(X_t\right) \quad , \quad \mathbb{E}_t\left[c_{t+J}^{-\sigma}\right] = \mathcal{N}\mathcal{N}_1^{(J)}\left(X_t\right)$$

$$\mathbb{E}_t\left[c_{t+1}^{-\sigma}\lambda_{t+1}\right] = \mathcal{N}\mathcal{N}_2^{(1)}\left(X_t\right) \quad , \quad \mathbb{E}_t\left[c_{t+J}^{-\sigma}\lambda_{t+J}\right] = \mathcal{N}\mathcal{N}_2^{(J)}\left(X_t\right)$$

$$\mathbb{E}_t\left[c_{t+J-1}^{-\sigma}\right] = \mathcal{N}\mathcal{N}_3^{(J)}\left(X_t\right)$$

If so, two equations from the system (27) would read:

$$\lambda_t = \frac{\mathbb{E}_t\left[\mathcal{N}\mathcal{N}_2^{(1)}\left(X_t\right)\right]}{\mathbb{E}_t\left[\mathcal{N}\mathcal{N}_1^{(1)}\left(X_t\right)\right]} \quad \text{and} \quad \lambda_t = \frac{\mathbb{E}_t\left[\mathcal{N}\mathcal{N}_2^{(J)}\left(X_t\right)\right]}{\mathbb{E}_t\left[\mathcal{N}\mathcal{N}_1^{(J)}\left(X_t\right)\right]}$$

Notice the the RHS of both equations is entirely comprised of pre-determined variables. The four-equation system (27) includes four unknowns and loses full-rank here.

One way to handle this issue is by using the forward states approach proposed in Faraglia et al. (2019). First, change the approximating functions so that endogenous variables show up on the RHS of each approximation:

$$E_t\left[c_t^{-\sigma}\right] = \mathcal{N}\mathcal{N}_1^{(1)}\left(X_t\right) \quad , \quad \mathbb{E}_t\left[c_{t+J-1}^{-\sigma}\right] = \mathcal{N}\mathcal{N}_1^{(J)}\left(X_t\right)$$

$$\mathbb{E}_t\left[c_t^{-\sigma}\lambda_t\right] = \mathcal{N}\mathcal{N}_2^{(1)}\left(X_t\right) \quad , \quad \mathbb{E}_t\left[c_{t+J-1}^{-\sigma}\lambda_{t+J-1}\right] = \mathcal{N}\mathcal{N}_2^{(J)}\left(X_t\right)$$

$$\mathbb{E}_t\left[c_{t+J-2}^{-\sigma}\right] = \mathcal{N}\mathcal{N}_3^{(J)}\left(X_t\right)$$

Then iterate forward and use the law of iterated expectations to write:

$$E_t\left[c_{t+1}^{-\sigma}\right] = \mathbb{E}_t\left[\mathcal{N}\mathcal{N}_1^{(1)}\left(X_{t+1}\right)\right] \quad , \quad \mathbb{E}_t\left[c_{t+J}^{-\sigma}\right] = \mathbb{E}_t\left[\mathcal{N}\mathcal{N}_1^{(J)}\left(X_{t+1}\right)\right]$$

$$\mathbb{E}_t\left[c_{t+1}^{-\sigma}\lambda_{t+1}\right] = \mathbb{E}_t\left[\mathcal{N}\mathcal{N}_2^{(1)}\left(X_{t+1}\right)\right] \quad , \quad \mathbb{E}_t\left[c_{t+J}^{-\sigma}\lambda_{t+J}\right] = \mathbb{E}_t\left[\mathcal{N}\mathcal{N}_2^{(J)}\left(X_{t+1}\right)\right]$$

$$\mathbb{E}_t\left[c_{t+J-1}^{-\sigma}\right] = \mathbb{E}_t\left[\mathcal{N}\mathcal{N}_3^{(J)}\left(X_{t+1}\right)\right]$$

Finally, use the properties of the Markov process to evaluate the RHS expectations.

Now the two equations from the system (27) that previously over-identified λ_t read:

$$\lambda_t = \frac{\mathbb{E}_t\left[\mathcal{N}\mathcal{N}_2^{(1)}\left(X_{t+1}\right)\right]}{\mathbb{E}_t\left[\mathcal{N}\mathcal{N}_1^{(1)}\left(X_{t+1}\right)\right]} \quad \text{and} \quad \lambda_t = \frac{\mathbb{E}_t\left[\mathcal{N}\mathcal{N}_2^{(J)}\left(X_{t+1}\right)\right]}{\mathbb{E}_t\left[\mathcal{N}\mathcal{N}_1^{(J)}\left(X_{t+1}\right)\right]}$$

Because $\boldsymbol{b}_t^{(t+j)}$, $\boldsymbol{b}_t^{(t+J)}$ and λ_t all appear on the RHS, the system returns to being non-degenerate.