Inference: Clustering

EC 607, Set 10

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Prologue

Schedule

Last time

Regression discontinuities

Today

Inference and clustering

Inference

Inference

Motivation

So far, we've focused on carefully **obtaining causal estimates** of the effect of some treatment $\mathbf{D}_i$ on our outcome $\mathbf{Y}_i$.

Our discussion of research designs and their requirements/assumptions has centered on **avoiding selection and securing unbiased and/or consistent estimates** for τ.

In other words, we've concentrated on **point estimates**.

What about **inference**?

Shminference [†]

Q Why care about inference?

A I'll give you two reasons.

1. We often want to **test theories/hypotheses**. Point estimates (*i.e.*, $\hat{\beta}$) can't do this alone. Inference finishes the job.

2. Other times, we want to ***measure* the effect** of a treatment. Inference helps us think about the **precision** of our estimates.

Note: Similar reasoning can apply to bounding forecasting/predictions.

If you want answers, then you need to do inference correctly.

† What is *shminference*?

What's so complicated?

Angrist and Pischke told us that "correcting" our standard errors for heteroskedasticity may increase the standard errors up to 25%.

What else are we worried about?

Inference

What we're worried about

- **Transformations of estimators**, *i.e.*, $\text{Var}\left[f\left(\hat{\beta}\right)\right] \neq f\left(\text{Var}\left[\hat{\beta}\right]\right)$

- **Dependence/correlation in our disturbance**, *i.e.*, $\text{Cov}\left(\varepsilon_i,\, \varepsilon_j\right) \neq 0$

  - Autocorrelation $\varepsilon_t = \rho\varepsilon_{t-1} + \varepsilon_t$
  - Correlated shocks within groups $\varepsilon_i = \varepsilon_{g(i)} + \varepsilon_i$

- **Finite-sample properties** *vs.* asymptotic properties

- **Power** and **minimal detectable effects**

- **Multiple-hypothesis testing** and ***p-hacking***

In other words: We've got a lot to worry/think about.

Clustering

Clustering

Setup

Many studies—observational and experimental—have a treatment that is assigned to all/most individuals within a group.

- Classrooms/schools
- Households
- Villages/counties/states

Furthermore, we might imagine individuals within the same group may have correlated disturbances. For i and j in group g

$$\text{Cov}\left(\varepsilon_i,\, \varepsilon_j\right) = E\left[\varepsilon_i \varepsilon_j\right] = \rho_\varepsilon \sigma_\varepsilon^2$$

where ρ_ε gives the within-group correlation of disturbances—what *MHE* calls the **intraclass correlation coefficient**.

Clustering

Setup

In other words, we have a regression

$$y_i = \beta_0 + \beta_1 x_{g(i)} + \varepsilon_i$$

where individual i is in group g, and $\mathbf{X}_{g(i)}$ only varies across groups.

For within-group correlation, we can use an additive random-effects model

$$\varepsilon_i = \nu_{g(i)} + \eta_i$$

meaning group members all receive a common shock $\nu_{g(i)}$, and individuals receive independent shocks η_i.

Note We assume η_i is independent of η_j $(i \neq j)$ and ν_g $(\forall g)$.

Clustering

Additive random effects

Based upon this model we've set up

$$\varepsilon_i = \nu_{g(i)} + \eta_i$$

the covariance between individuals i and j in group g is

$$\begin{aligned}\text{Cov}\left(\varepsilon_i,\, \varepsilon_j\right) &= E\left[\varepsilon_i \varepsilon_j\right] = E\left[\left(\nu_g + \eta_i\right)\left(\nu_g + \eta_j\right)\right] = E\left[\nu_g^2\right] = \sigma_\nu^2 \\ &= \rho_\varepsilon \sigma_\varepsilon^2 \\ &= \rho_\varepsilon \left(\sigma_\nu^2 + \sigma_\eta^2\right)\end{aligned}$$

Thus, we can write the intraclass correlation coefficient as

$$\rho_\varepsilon = \frac{\sigma_\nu^2}{\sigma_\varepsilon^2} = \frac{\sigma_\nu^2}{\sigma_\nu^2 + \sigma_\eta^2}$$

Clustering

What is ρ_ε?

Let's review what we know.

$$\varepsilon_i = \nu_{g(i)} + \eta_i \qquad \text{and} \qquad \rho_\varepsilon = \frac{\sigma_\nu^2}{\sigma_\varepsilon^2} = \frac{\sigma_\nu^2}{\sigma_\nu^2 + \sigma_\eta^2}$$

One way to think about ρ_ε is as the **share of the variance of the disturbance ε_i accounted for by the shared disurbance** $\nu_{g(i)}$.

As $\nu_{g(i)}$ accounts for more and more of the variation in ε_i, $\rho_\varepsilon \to 1$.

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So...

Q Why do we care about ρ_ε?

A It tells us by how wrong our standard errors can be if we treat all observations as independent.

Let $\mathbf{Var}_o\left(\hat{\beta}_1\right)$ denote the conventional variance formula for OLS estimator.†

Let $\mathbf{Var}\left(\hat{\beta}_1\right)$ denote the actual variance of $\hat{\beta}_1$.

† which treats all disturbances as independent (and identically distributed).

Clustering

So....

With (**1**) nonstochastic regressors fixed by group *and* (**2**) groups of size n

$$\frac{\mathbf{Var}\left(\hat{\beta}_1\right)}{\mathbf{Var}_o\left(\hat{\beta}_1\right)} = 1 + (n-1)\rho_\varepsilon \quad \Longrightarrow \quad \frac{\mathbf{S.E.}\left(\hat{\beta}_1\right)}{\mathbf{S.E.}_o\left(\hat{\beta}_1\right)} = \sqrt{1 + (n-1)\rho_\varepsilon}$$

The term $\sqrt{1 + (n-1)\rho_\varepsilon}$ is called the Moulton factor[†].

The Moulton factor tells us by what factor standard errors will be wrong if we ignore within-group correlation (conditional on assumptions **1** and **2**).

Q What happens if $\rho = 1$? What if you duplicated your dataset?

Q What happens as n increases?

† After Moulton (1986). Derivation: *MHE* 323–325.

The Moulton factor

The Moulton factor

$$\frac{\text{S.E.}\left(\hat{\beta}_1\right)}{\text{S.E.}_o\left(\hat{\beta}_1\right)} = \sqrt{1 + (n-1)\rho_\varepsilon}$$

shows even when ρ_ε is small, we can have vary large standard error issues.

Ex An experiment on 400 schools, each with 1,000 students.

If $\rho_\varepsilon = \mathbf{0.01}$, the Moulton factor is $\sqrt{1 + (1,000 - 1) \times \mathbf{0.01}} \approx \mathbf{3.32}$.

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Test statistics

Recall $t_{\text{stat}} = \dfrac{\hat{\beta}_1}{\text{S.E.}\left(\hat{\beta}_1\right)}$.

$$\therefore \frac{t_o}{t} = \frac{\hat{\beta}_1 / \text{S.E.}_o\left(\hat{\beta}_1\right)}{\hat{\beta}_1 / \text{S.E.}\left(\hat{\beta}_1\right)} = \frac{\text{S.E.}\left(\hat{\beta}_1\right)}{\text{S.E.}_o\left(\hat{\beta}_1\right)} = \text{the Moulton factor.}$$

Ex Thus, in our example of 400 schools with 1,000 students, ignoring within-school correlation of $\rho_\varepsilon = 0.01$ would lead us test statistics that are more than 3 times as large as they should be.

This is why economics seminars have standard-error police.

Clustering

Relaxing assumptions

If we allow regressors to vary by individual and groups to differ in size (n_g),

$$\frac{\mathrm{Var}\left(\hat{\beta}_1\right)}{\mathrm{Var}_o\left(\hat{\beta}_1\right)} = 1 + \left[\frac{\mathrm{Var}(n_g)}{\overline{n}} + \overline{n} - 1\right]\rho_x\rho_\varepsilon$$

where ρ_x denotes the intraclass (within-group) correlation of x_i.†

Important The Moulton factor for this general model depends upon the amount of within-group correlation in x_i *and* ε_i.

The special case is also important, as treatment is often fixed at some level.

† See *MHE* for mathematical definitions and the derivation.

Clustering

The answer

Q So what do we do now?

A We've got options (as usual)

1. Parametrically model the random effects
2. Cluster-robust standard error (estimator)
3. Aggregate up to the group (or a similar method)
4. Block (group-based) bootstrap
5. GLS/MLE modeling y_i and ε_i

Most common: Cluster-robust standard errors

Runner up: Block bootstrap

Second runner up: Group-level analysis

Clustering

Cluster-robust standard errors

Liang and Zeger (1986) extend White's heteroskedasticity-robust covariance matrix to allow for both clustering and heteroskedasticity.[†]

$$\hat{\Omega}_{\text{cl}} = (\mathrm{X}'\mathrm{X})^{-1} \left(\sum_g \mathrm{X}_g' \hat{\Psi}_g \mathrm{X}_g \right) (\mathrm{X}'\mathrm{X})^{-1}$$

$$\hat{\Psi}_g = a e_g e_g' = a \begin{bmatrix} e_{1g}^2 & e_{1g}e_{2g} & \cdots & e_{1g}e_{n_g g} \\ e_{1g}e_{2g} & e_{2g}^2 & e_{2g}\cdots & e_{2g}e_{n_g g} \\ \vdots & \vdots & \ddots & \vdots \\ e_{1g}e_{n_g g} & e_{2g}e_{n_g g} & \cdots & e_{n_g g}^2 \end{bmatrix}$$

where e_g are the OLS residuals for group g, e_{ig} is the residual for individual i in group g, and a is a degrees-of-freedom adjustment.

† When people say *clustering*, they typically mean *correlated disturbances within a group*.

Clustering

Cluster-robust standard errors

Derivation Let $\mathbf{x}_i$ denote observation i (row) from $\mathbf{X}$.

$$\text{Var}\left(\hat{\beta}\middle|\mathbf{X}\right) = E\left[\left(\hat{\beta}-\beta\right)\left(\hat{\beta}-\beta\right)'\middle|\mathbf{X}\right] = E\left[\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\varepsilon\varepsilon'\mathbf{X}\left(\mathbf{X}'\mathbf{X}\right)^{-1}\middle|\mathbf{X}\right]$$

$$= \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\,E\left[\varepsilon\varepsilon'\middle|\mathbf{X}\right]\mathbf{X}\left(\mathbf{X}'\mathbf{X}\right)^{-1}$$

$$= \left(\sum_{i=1}^{N}\mathbf{x}_i'\mathbf{x}_i\right)^{-1}\left(\sum_{i=1}^{N}\sum_{j=1}^{N}\mathbf{x}_i'\mathbf{x}_j\,E\left[\varepsilon_j\varepsilon_i\middle|\mathbf{X}\right]\right)\left(\sum_{i=1}^{N}\mathbf{x}_i'\mathbf{x}_i\right)^{-1}$$

Q Can we estimate $\left(\sum_i\sum_j\mathbf{x}_i'\mathbf{x}_j\,E\left[\varepsilon_j\varepsilon_i\middle|\mathbf{X}\right]\right)$ with $\sum_i\sum_j\mathbf{x}_i'\mathbf{x}_j e_j e_i = \mathbf{X}'ee'\mathbf{X}$?

A No. Recall with OLS, $\mathbf{X}'e = \mathbf{0}$. But we will do something similar.

Cluster-robust standard errors

Imagine we have G clusters with some unknown dependence between observations within a cluster and independence between clusters.

Then we can ignore $\mathbf{x}_i'\mathbf{x}_j\, E\left[\varepsilon_j\varepsilon_i \middle| \mathbf{X}\right]$ if i and j are in different clusters.

We can estimate $\sum_i \sum_j \mathbf{x}_i'\mathbf{x}_j\, E\left[\varepsilon_j\varepsilon_i \middle| \mathbf{X}\right]$ with

$$\sum_{g=1}^{G} \left(\sum_{i=1}^{N_g} \sum_{j=1}^{N_g} \mathbf{x}_i'\mathbf{x}_j e_j e_i \right) = \sum_{g=1}^{G} \mathbf{X}_g' e_g e_g' \mathbf{X}_g$$

I.e., to learn about within-group covariance, we calculate these within-group cross products and then sum over groups.†

† Group sizes can vary.

Clustering

Guidelines for group number/size

Large G, **Small** N_g

Clustered standard errors work well. $G > N_g$ and $G > 20$.

Large G, **Large** N_g

We might be concerned about the number of within-group cross terms here. However, for moderately large G (50?), cluster-robust standard errors appear to perform well with large N_g.

Small G, **Large** N_g

Cluster-robust standard errors do not work well (definitely $G < 10$).

Options Collapse groups? Wild clustered bootstrap?

Small G, **Small** N_g

Essentially the same issues and solutions as small G with large N_g.

Clustering

Further extensions

We've discussed the standard cluster-robust variance-covariance estimator.

Multi-way clustering allows multiple levels/dimensions in which individuals are *clustered*.

- For *nested clusters* (*e.g.*, state and county), people commonly cluster at the highest (largest) unit.

- For *non-nested clusters* (*e.g.*, state and year), Cameron, Gelbach, and Miller (2011) provide a covariance estimator

$$\mathbf{Var}\left(\hat{\beta}\right) = \mathbf{Var}_{\text{State}}\left(\hat{\beta}\right) + \mathbf{Var}_{\text{Year}}\left(\hat{\beta}\right) - \mathbf{Var}_{\text{State-Year}}\left(\hat{\beta}\right)$$

  where $\mathbf{Var}_{\text{State}}\left(\hat{\beta}\right)$ denotes the covariance of $\hat{\beta}$ clustered by state.

Clustering

Further extensions

We've discussed the standard cluster-robust variance-covariance estimator.

The term **Conley standard errors** is often used to describe situations in which you have spatial clustering/correlation that you can describe via a function like spatial distance.†

See Conley (1999) for the paper and this blog by Dan Christensen and Thiemo Fetzer for practical implementation in `R` and `Stata`.

† They also are robust to heteroskedasticity and autocorrelation within units.

Cluster-robust standard errors

So now you know what `lm_robust()`, `iv_robust()`, *etc.* are doing when you specify a variable for clustering (*e.g.*, `clusters = var`).

`lm_robust()` without clustering

```
# Estimate without clusters
vote_no <- lm_robust(
  voteA ~ expendA + expendB,
  fixed_effects = state,
  data = wooldridge::vote1
)
```

`lm_robust()` with clustering

```
# Estimate with clusters
vote_cl <- lm_robust(
  voteA ~ expendA + expendB,
  fixed_effects = state,
  clusters = state,
  data = wooldridge::vote1
)
```

Clustering

Cluster-robust standard errors

Alternatives for clustering: `felm()` from `lfe` and `feols()` from `fixest`.

`felm()` clustering by state

```
# Estimate with clusters
est_felm = felm(
  voteA ~ expendA + expendB |
  state |
  0 |
  state,
  data = wooldridge::vote1
)
```

`feols()` clustering by state

```
# Estimate with clusters
est_feols = feols(
  voteA ~ expendA + expendB |
  state,
  data = wooldridge::vote1
)
# Force cluster-rob. SEs
summary(
  est_feols,
  se = "cluster",
  cluster = "state"
)
```

Time for a simulation.

Cluster simulation

Cluster simulation

The DGP

Let's opt for a simple-ish example.†

$$y_{ig} = (\beta_0 = 1) + (\beta_1 = 2)\, x_{1,g} + (\beta_2 = 0)\, x_{2,g} + \varepsilon_{ig}$$
$$\varepsilon_{ig} = \nu_g + \eta_i$$

where the $\eta_i \perp \eta_j$, $\eta_i \perp \nu_g$, and $\nu_g \perp \nu_h$.

Let's assume $\eta_i \sim N(0, 1)$ and $\nu_g \sim N(0, 1)$. And $x_g \sim N(0, 1)$.

Plus $N_g = 100$ with 10 groups.

Note Small G with large-ish N_g.

† So we have more room for problem sets/exams.

First we need to write the **data generating process for one iteration**.

```
# The DGP
sim_dgp <- function(n = 100, n_grps = 10, σv = 1, ση = 1) {
  # Create the right number of observations
  sample_df <- expand.grid(i = 1:n, g = 1:n_grps) %>% as_tibble()
  # Create a unique ID (from 1 to number of observations)
  sample_df %<>% mutate(id = 1:(n * n_grps))
  # Sample v at the group level (NOTE: DON'T FORGET TO UNGROUP)
  sample_df %<>% group_by(g) %>%
    mutate(v = rnorm(1, sd = σv)) %>% ungroup()
  # Sample η at the individual level
  sample_df %<>% mutate(η = rnorm(n * n_grps, sd = ση))
  # Sample x_g from N(0,1)
  sample_df %<>% group_by(g) %>%
    mutate(x1 = rnorm(1), x2 = rnorm(1)) %>% ungroup()
  # Calculate y
  sample_df %<>% mutate(y = 1 + 2 * x1 + 0 * x2 + v + η)
  # Return
  return(sample_df)
}
```

Now we **analyze** the data within one iteration.

```r
# Analyze 'data'
sim_analyze ← function(data) {
  # Conventional SEs
  result_ols ← lm_robust(
    y ~ x1 + x2, data = data, se_type = "classical"
  ) %>% tidy() %>% filter(term %in% c("x1", "x2")) %>% select(1:5) %>%
  mutate(type = "conventional")
  # Cluster-robust SEs
  result_cl ← lm_robust(
    y ~ x1 + x2, data = data, clusters = g
  ) %>% tidy() %>% filter(term %in% c("x1", "x2")) %>% select(1:5) %>%
  mutate(type = "clustered")
  # Bind results together and add column for standard errors
  results_df ← bind_rows(result_ols, result_cl)
  # Return results
  return(results_df)
}
```

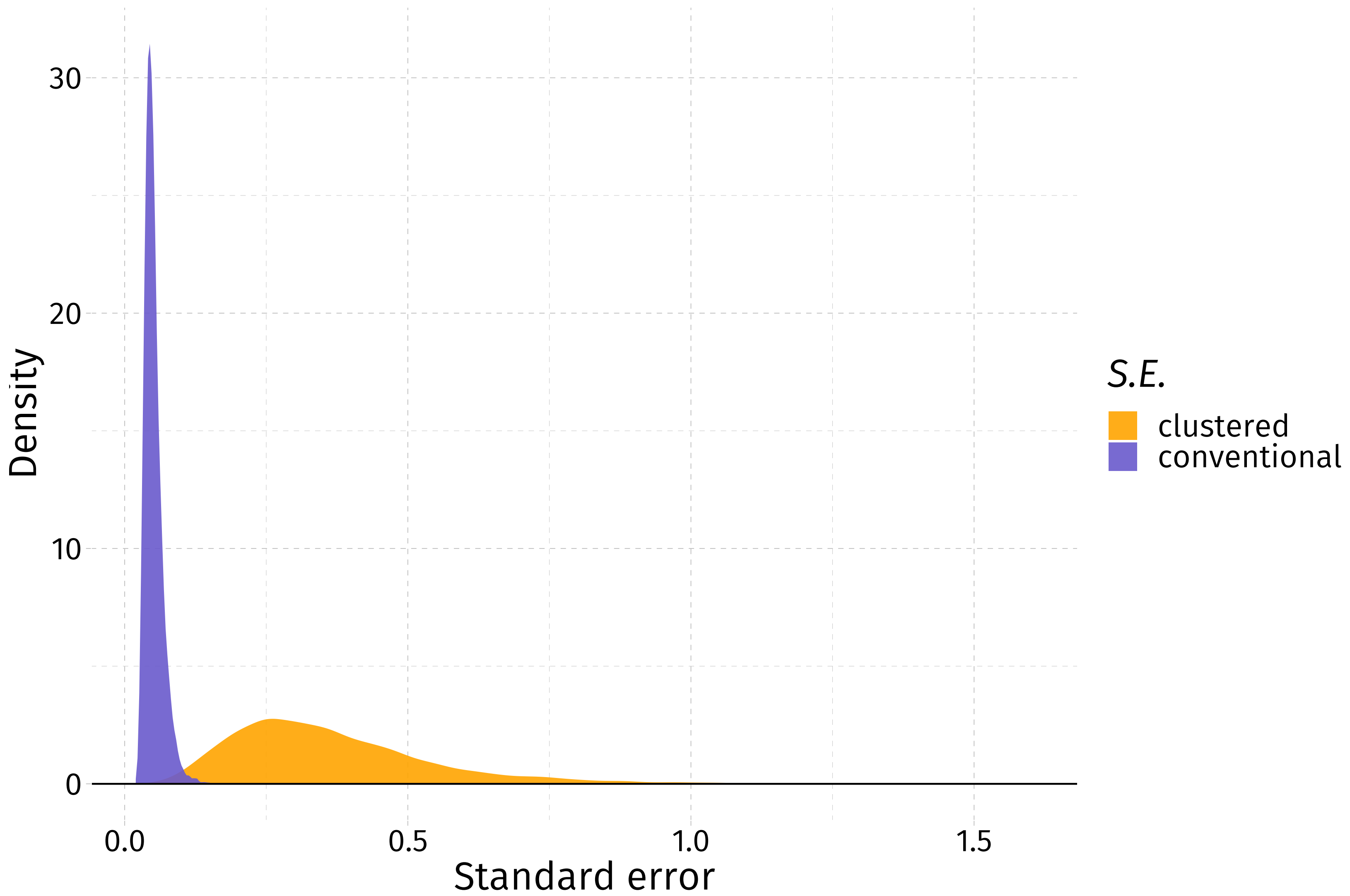
Now put the pieces together.

```r
# Join sim_dgp and sim_analyze
sim_iter <- function(n = 100, n_grps = 10, σν = 1, ση = 1) {
  # Run the analysis in sim_analyze on the output of sim_dgp
  sim_dgp(n = 100, n_grps = 10, σν = 1, ση = 1) %>% sim_analyze()
}
```

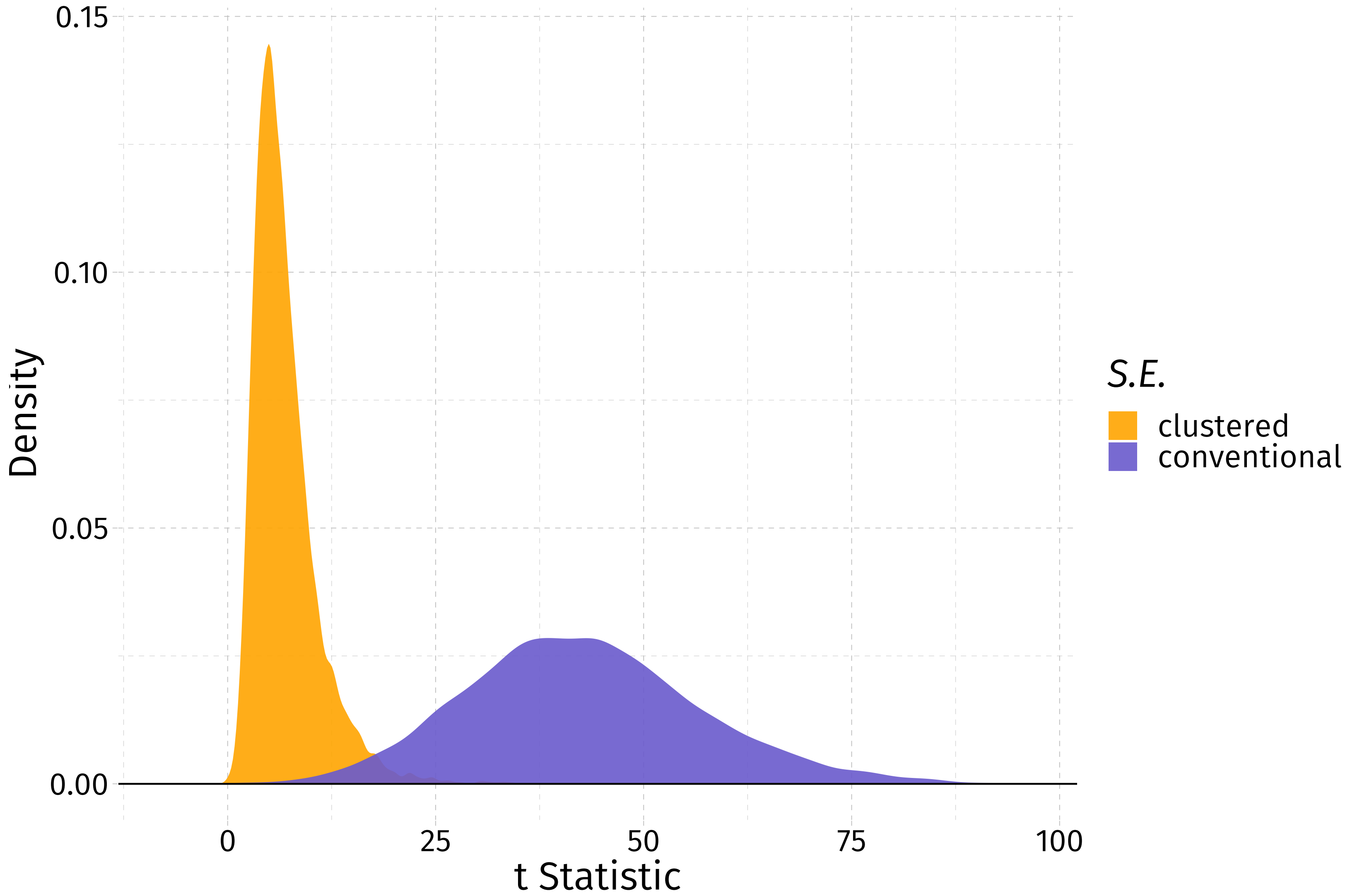
And we **run the simulation** (10,000 times).

```
# Load and set up furrr
p_load(furrr)
plan(multiprocess, workers = 10)
# Set a seed
set.seed(1234)
# Run the simulation 1e4 times
sim_df <- future_map_dfr(
  # Repeat sample size 100 for 1e4 times
  rep(100, 1e4),
  # Our function
  sim_iter,
  # Let furrr know we want to set a seed
  .options = future_options(seed = T)
)
```

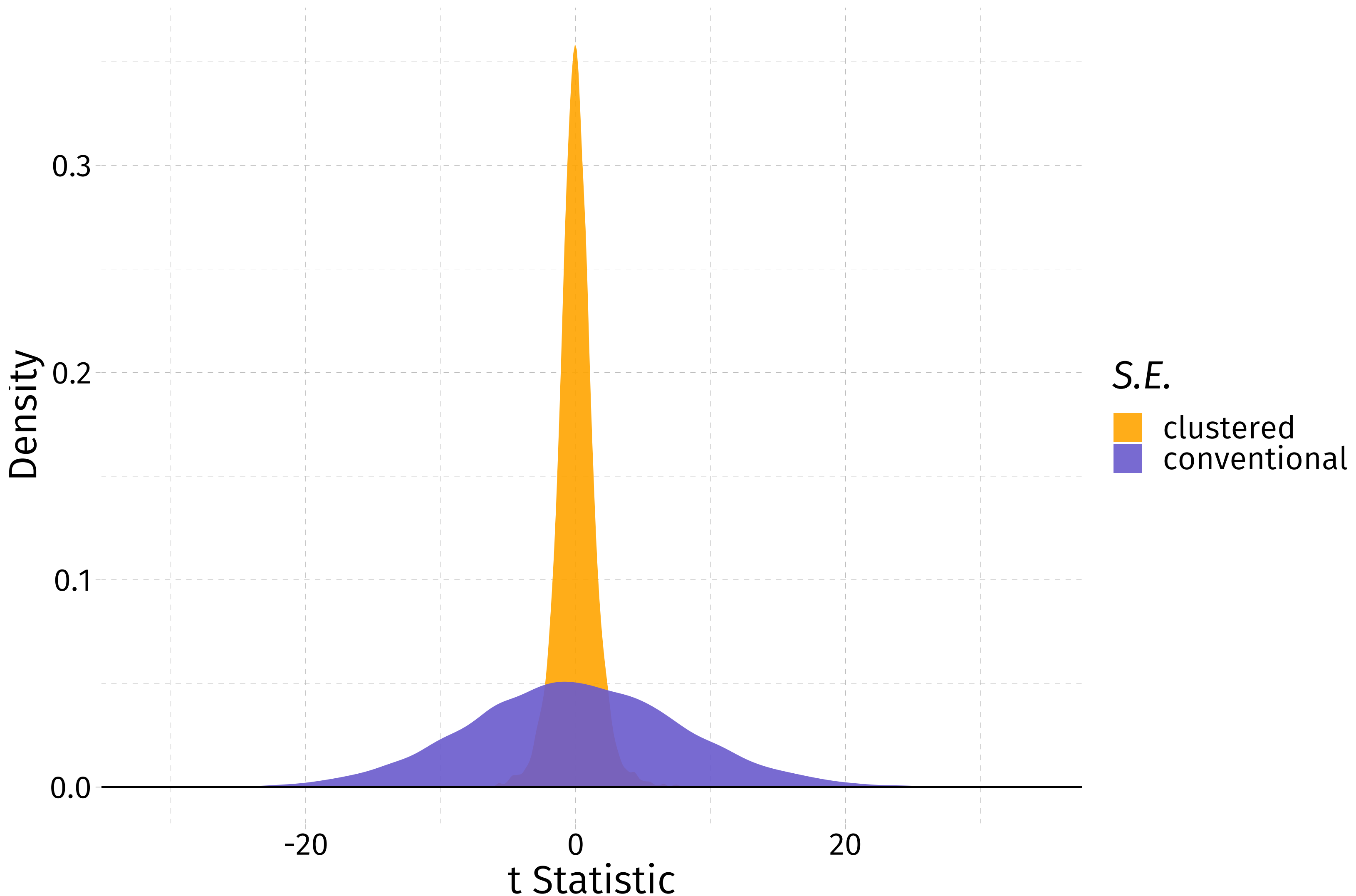
Comparing standard errors for $\hat{\beta}_1$ (coefficient on x_1)



Comparing *t* statistics for $\hat{\beta}_1$ (coefficient on x_1)



Comparing *t* statistics for $\hat{\beta}_2$ (coefficient on x_2)



Rejection rates

x1	clustered	0.878
x1	conventional	0.999
x2	clustered	0.0371
x2	conventional	0.801

1. We definitely can see the **need for clustering**.
   Conventional standard errors are rejecting a true H_0 80% of the time.

2. **Cluster-robust standard errors are struggling** a bit in this situation.
   Small G; large N_g. Rejecting false H_0 88% and true H_0 3.7% of the time.

Resources from the literature

When Should You Adjust Standard Errors for Clustering?
Abadie, Athey, Imbens, and Wooldridge

A Practitioner's Guide to Cluster-Robust Inference
Cameron and Miller (2015)

Robust Inference With Multiway Clustering
Cameron, Gelbach, and Miller (2011)

Bootstrap-Based Improvements for Inference with Clustered Errors
Cameron, Gelbach, and Miller (2008)

How Much Should We Trust Differences-In-Differences Estimates?
Bertrand, Duflo, and Mullainathan (2004)

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