Inference and Simulation

EC 607, Set 04

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Spring 2020

Prologue

Schedule

Last time

The *CEF* and least-squares regression

Today

Inference

Read *MHE* 3.1

Upcoming

Lab (as usual) on Friday.

Inference

Inference

Why?

Q What's the big deal with inference?

A We rarely know the CEF or the population (and its regression vector).

We *can* draw statistical inferences about the population using samples.

Important The issue/topic of *statistical inference* is separate from *causality*.

Separate questions

1. How do we interpret the estimated coefficient $\hat{\beta}$?
2. What is the sampling distribution of $\hat{\beta}$?

Inference

Moving from population to sample

Recall The population-regression function gives us the best linear approximation to the CEF.

We're interested in the (unknown) population-regression vector

$$\beta = \boldsymbol{E}\left[\mathbf{X}_i \mathbf{X}_i'\right]^{-1} \boldsymbol{E}[\mathbf{X}_i \mathbf{Y}_i]$$

which we estimate via the ordinary least squares (OLS) estimator†

$$\hat{\beta} = \left(\sum_i \mathbf{X}_i \mathbf{X}_i' \right)^{-1} \left(\sum_i \mathbf{X}_i \mathbf{Y}_i \right)$$

† *MHE* presents a method-of-moments motivation for this derivation, where $\dfrac{1}{n}\sum_i \mathbf{X}_i \mathbf{X}_i'$ is our sample-based estimated for $\boldsymbol{E}\left[\mathbf{X}_i \mathbf{X}_i'\right]$. You've also seen others, *e.g.*, minimizing MSE of $\mathbf{Y}_i$ given $\mathbf{X}_i$.

Inference

A classic

However you write it, this OLS estimator

$$
\begin{aligned}
\hat{\beta} &= (\mathrm{X}'\mathrm{X})^{-1}\mathrm{X}'\mathrm{y} \\
&= \left(\sum_i \mathrm{X}_i\mathrm{X}_i'\right)^{-1}\left(\sum_i \mathrm{X}_i\mathrm{Y}_i\right) \\
&= \beta + \left[\sum_i \mathrm{X}_i\mathrm{X}_i'\right]^{-1}\sum_i \mathrm{X}_i e_i
\end{aligned}
$$

is the same estimator you've been using since undergrad.

Note I'm following *MHE* in defining $e_i = \mathbf{Y}_i - \mathbf{X}_i'\beta$.

Inference

A classic

As you've learned, the OLS estimator

$$\hat{\beta} = \left(\sum_i \mathrm{X}_i \mathrm{X}_i'\right)^{-1} \left(\sum_i \mathrm{X}_i \mathrm{Y}_i\right) = \beta + \left[\sum_i \mathrm{X}_i \mathrm{X}_i'\right]^{-1} \sum_i \mathrm{X}_i e_i$$

has asymptotic covariance

$$\boldsymbol{E}\left[\mathrm{X}_i \mathrm{X}_i'\right]^{-1} \boldsymbol{E}\left[\mathrm{X}_i \mathrm{X}_i' e_i^2\right] \boldsymbol{E}\left[\mathrm{X}_i \mathrm{X}_i'\right]^{-1}$$

which we estimate by (**1**) replacing e_i with $\hat{e}_i = \mathrm{Y}_i - \mathrm{X}_i'\hat{\beta}$ and (**2**) replacing expectations with sample means, *e.g.*, $\boldsymbol{E}\left[\mathrm{X}_i \mathrm{X}_i' e_i^2\right]$ becomes $\dfrac{1}{n}\sum\left[\mathrm{X}_i \mathrm{X}_i' \hat{e}_i^2\right]$.

Standard errors of this flavor are known as heteroskedasticity-consistent (or -robust) standard errors (or Eicker-Huber-White).

Inference

Defaults

Statistical packages default to assuming homoskedasticity, *i.e.*, $E\big[e_i^2 \mid \mathrm{X}_i\big] = \sigma^2$ for all i. With homoskedasticity,

$$E\big[\mathrm{X}_i\mathrm{X}_i' e_i^2\big] = E\big[E\big[\mathrm{X}_i\mathrm{X}_i' e_i^2 \mid \mathrm{X}_i\big]\big] = E\big[\mathrm{X}_i\mathrm{X}_i'\, E\big[e_i^2 \mid \mathrm{X}_i\big]\big] = \sigma^2\, E\big[\mathrm{X}_i\mathrm{X}_i'\big]$$

Now, returning to to the asym. covariance matrix of $\hat{\beta}$,

$$\begin{aligned} E\left[\mathrm{X}_i\mathrm{X}_i'\right]^{-1} E\big[\mathrm{X}_i\mathrm{X}_i' e_i^2\big]\, E\left[\mathrm{X}_i\mathrm{X}_i'\right]^{-1} &= E\left[\mathrm{X}_i\mathrm{X}_i'\right]^{-1} \sigma^2\, E\big[\mathrm{X}_i\mathrm{X}_i'\big]\, E\left[\mathrm{X}_i\mathrm{X}_i'\right]^{-1} \\ &= \sigma^2\, E\left[\mathrm{X}_i\mathrm{X}_i'\right]^{-1} \end{aligned}$$

Inference

Defaults

Angrist and Pischke argue we should probably change our default to heteroskedasticity.

If the CEF is nonlinear, then our linear approximation (linear regression) generates heteroskedasticity.

$$\begin{aligned}
&E\Big[\big(\mathrm{Y}_i - \mathrm{X}_i'\beta\big)^2 \mid \mathrm{X}_i\Big] \\
&\quad = E\Bigg[\bigg(\big\{\mathrm{Y}_i - E[\mathrm{Y}_i \mid \mathrm{X}_i]\big\} + \big\{\,E[\mathrm{Y}_i \mid \mathrm{X}_i] - \mathrm{X}_i'\beta\big\}\bigg)^2 \bigg| \mathrm{X}_i\Bigg] \\
&\quad = \mathrm{Var}(\mathrm{Y}_i \mid \mathrm{X}_i) + \big(E[\mathrm{Y}_i \mid \mathrm{X}_i] - \mathrm{X}_i'\beta\big)^2
\end{aligned}$$

Thus, even if $\mathbf{Y}_i \mid \mathbf{X}_i$ has contant variance, $e_i \mid \mathbf{X}_i$ is heteroskedastic.

Inference

Two notes

1. Heteroskedasticity is **not our biggest concern** in inference.

> ...as an empirical matter, heteroskedasticity may matter very little... If heteroskedasticity matters a lot, say, more than a 30 percent increase or any marked decrease in standard errors, you should worry about possible programming errors or other problems. (*MHE*, p.47)

2. Notice that we've **avoided "standard" stronger assumptions**, *e.g.*, normality, fixed regressors, linear CEF, homoskedasticity.

Following (2): We only have large-sample, asymptotic results (consistency) rather than finite-sample results (unbiasedness).

Inference

Warning

Because many of properties we care about for the inference are **large-sample** properties, they may not always apply to **small samples**.

One practical way we can study the behavior of an estimator: **simulation**.

Note You need to make sure your simulation can actually test/respond to the question you are asking (*e.g.*, bias *vs.* consistency).

Inference

Simulation

Let's compare false- and true-positive rates[†] for

1. **Homoskedasticity-assuming standard errors** $\left(\text{Var}[e_i|\text{X}_i] = \sigma^2\right)$
2. **Heteroskedasticity-robust standard errors**

Simulation outline

```
1. Define data-generating process (DGP).
2. Choose sample size n.
3. Set seed.
4. Run 10,000 iterations of
       a.  Draw sample of size n from DGP.
       b.  Conduction inference.
       c.  Record inferences' outcomes.
```

† The false-positive rate goes by many names; another common name: *type-I error rate*.

Simulation

Simulation

Data-generating process

First, we'll define our DGP.

We've been talking a lot about nonlinear CEFs, so let's use one.

Let's keep the disturbances well behaved.

$$\mathbf{Y}_i = 1 + e^{0.5\mathbf{X}_i} + \varepsilon_i$$

where $\mathbf{X}_i \sim \text{Uniform}(0, 10)$ and $\varepsilon_i \sim N(0, 1)$.

Simulation

Data-generating process

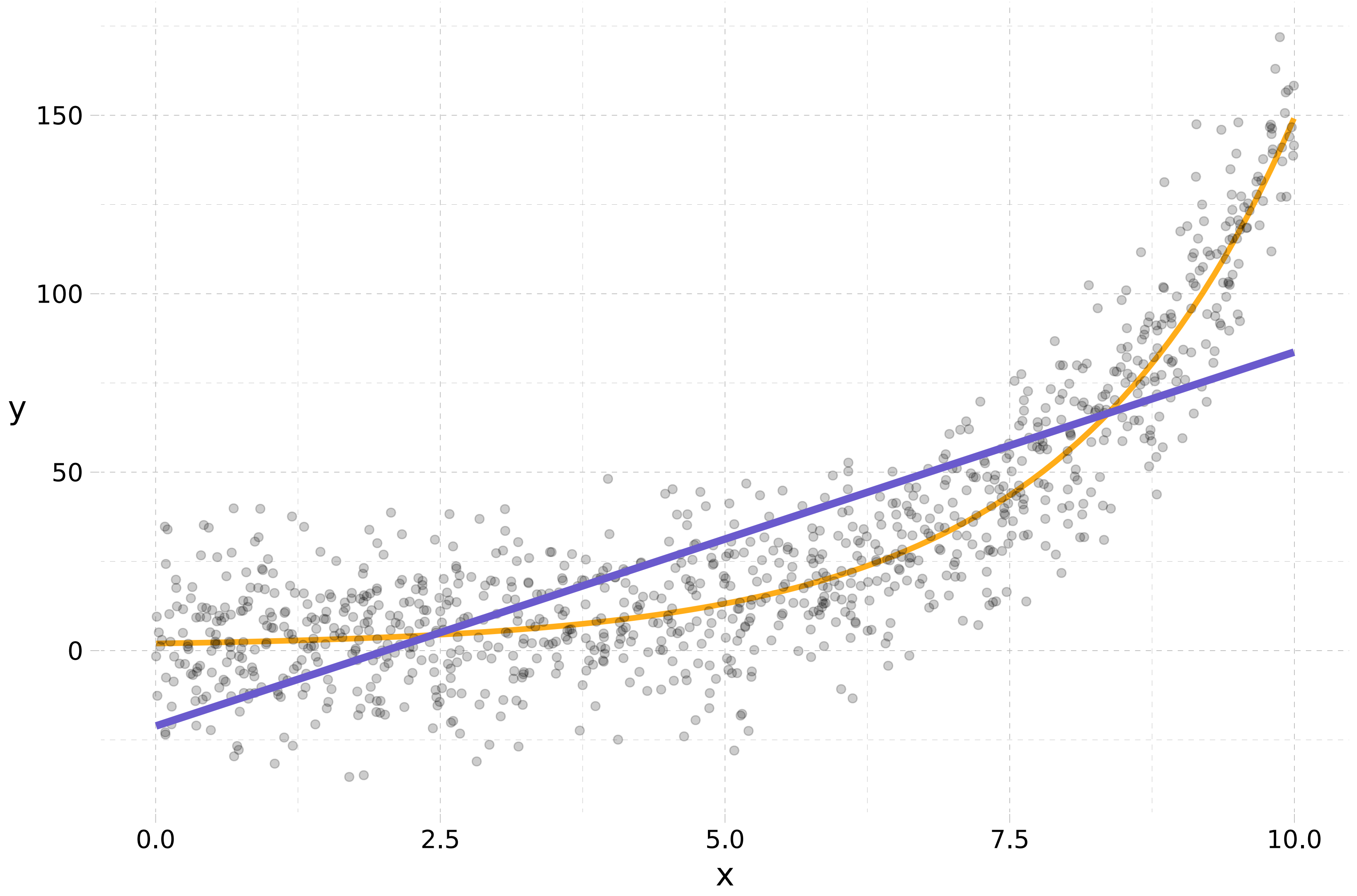
$$\text{Y}_i = 1 + e^{0.5 \text{X}_i} + \varepsilon_i$$

where $\text{X}_i \sim \mathrm{Uniform}(0, 10)$ and $\varepsilon_i \sim N(0, 15^2)$.

```r
library(pacman)
p_load(dplyr)
# Choose a size
n ← 1000
# Generate data
dgp_df ← tibble(
  ε = rnorm(n, sd = 15),
  x = runif(n, min = 0, max = 10),
  y = 1 + exp(0.5 * x) + ε
)
```

```
#> # A tibble: 1,000 x 3
#>          ε     x      y
#>      <dbl> <dbl>  <dbl>
#>  1    8.78  9.53 127.
#>  2   10.6   6.22  34.0
#>  3   -1.64  5.32  13.6
#>  4   -6.80  8.92  80.7
#>  5    9.09  1.96  12.8
#>  6  -27.3   8.84  57.0
#>  7    9.45  2.18  13.4
#>  8   -4.14  3.78   3.47
#>  9   -4.26  3.52   2.54
#> 10  -13.8   9.88 127.
#> # … with 990 more rows
```

The population least-squares regression line



Simulation

Iterating

To make iterating easier, let's wrap our DGP in a function.

```r
fun_iter ← function(iter, n = 30) {
  # Generate data
  iter_df ← tibble(
    ε = rnorm(n, sd = 15),
    x = runif(n, min = 0, max = 10),
    y = 1 + exp(0.5 * x) + ε
  )
}
```

We still need to run a regression and draw some inferences.

Note We're defaulting to size-30 samples.

Simulation

We will use `lm_robust()` from the `estimatr` package for OLS and inference.[†]

- `se_type = "classical"` provides homoskedasticity-assuming SEs
- `se_type = "HC2"` provides heteroskedasticity-robust SEs

```r
lm_robust(y ~ x, data = dgp_df, se_type = "classical") %>% tidy() %>% select(1:5)
```

```
#>          term  estimate std.error statistic       p.value
#> 1 (Intercept) -21.14183  1.473496 -14.34807  1.383951e-42
#> 2           x  10.48074  0.257810  40.65294 6.560626e-214
```

```r
lm_robust(y ~ x, data = dgp_df, se_type = "HC2") %>% tidy() %>% select(1:5)
```

```
#>          term  estimate std.error statistic       p.value
#> 1 (Intercept) -21.14183 1.4335274 -14.74812  1.112039e-44
#> 2           x  10.48074 0.3097606  33.83495 8.788638e-168
```

† `lm()` works for "spherical" standard errors but cannot calculate het.-robust standard errors.

Simulation

Inference

Now add these estimators to our iteration function...

```r
fun_iter <- function(iter, n = 30) {
  # Generate data
  iter_df <- tibble(
    ε = rnorm(n, sd = 15),
    x = runif(n, min = 0, max = 10),
    y = 1 + exp(0.5 * x) + ε
  )
  # Estimate models
  lm1 <- lm_robust(y ~ x, data = iter_df, se_type = "classical")
  lm2 <- lm_robust(y ~ x, data = iter_df, se_type = "HC2")
  # Stack and return results
  bind_rows(tidy(lm1), tidy(lm2)) %>%
    select(1:5) %>% filter(term == "x") %>%
    mutate(se_type = c("classical", "HC2"), i = iter)
}
```

Simulation

Run it

Now we need to actually run our `iter_df()` function 10,000 times.

There are a lot of ways to run a single function over a list/vector of values.

- `lapply()`, *e.g.*, `lapply(X = 1:3, FUN = sqrt)`
- `for()`, *e.g.*, `for (x in 1:3) sqrt(x)`
- `map()` from `purrr`, *e.g.*, `map(1:3, sqrt)`

We're going to go with `map()` from the `purrr` package because it easily parallelizes across platforms using the `furrr` package.

Simulation

Run it!

Run our function 10,000 times

```
# Packages
p_load(purrr)
# Set seed
set.seed(12345)
# Run 10,000 iterations
sim_list ← map(1:1e4, fun_iter)
```

Parallelized 10,000 iterations

```
# Packages
p_load(purrr, furrr)
# Set options
set.seed(123)
# Tell R to parallelize
plan(multiprocess)
# Run 10,000 iterations
sim_list ← future_map(
  1:1e4, fun_iter,
  .options = future_options(seed = T)
)
```

The `furrr` package (`future` + `purrr`) makes parallelization **easy and fun**!😸

Simulation

Run it!!

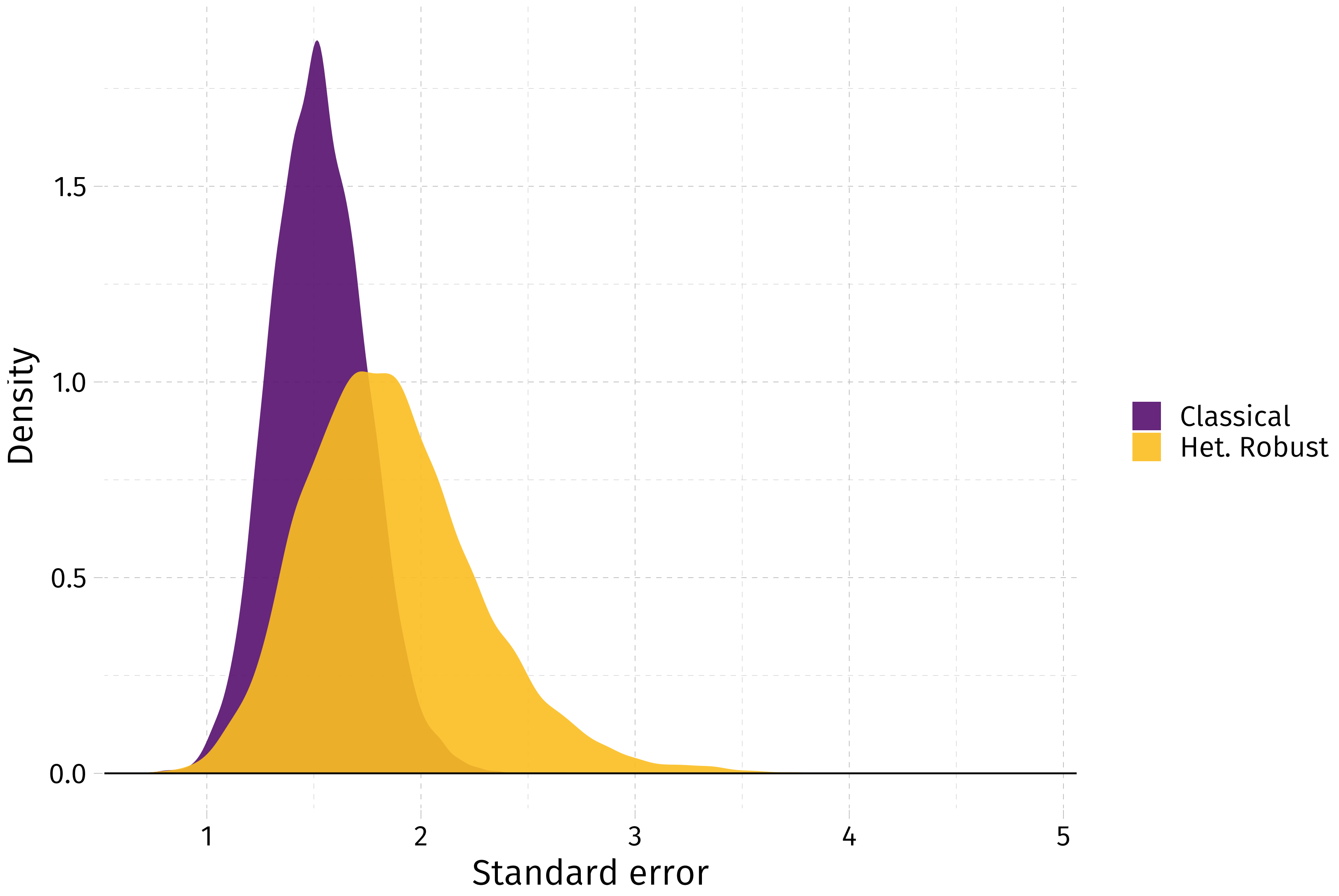
Our `fun_iter()` function returns a `data.frame`, and `future_map()` returns a `list` (of the returned objects).

So `sim_list` is going to be a `list` of `data.frame` objects. We can bind them into one `data.frame` with `bind_rows()`.

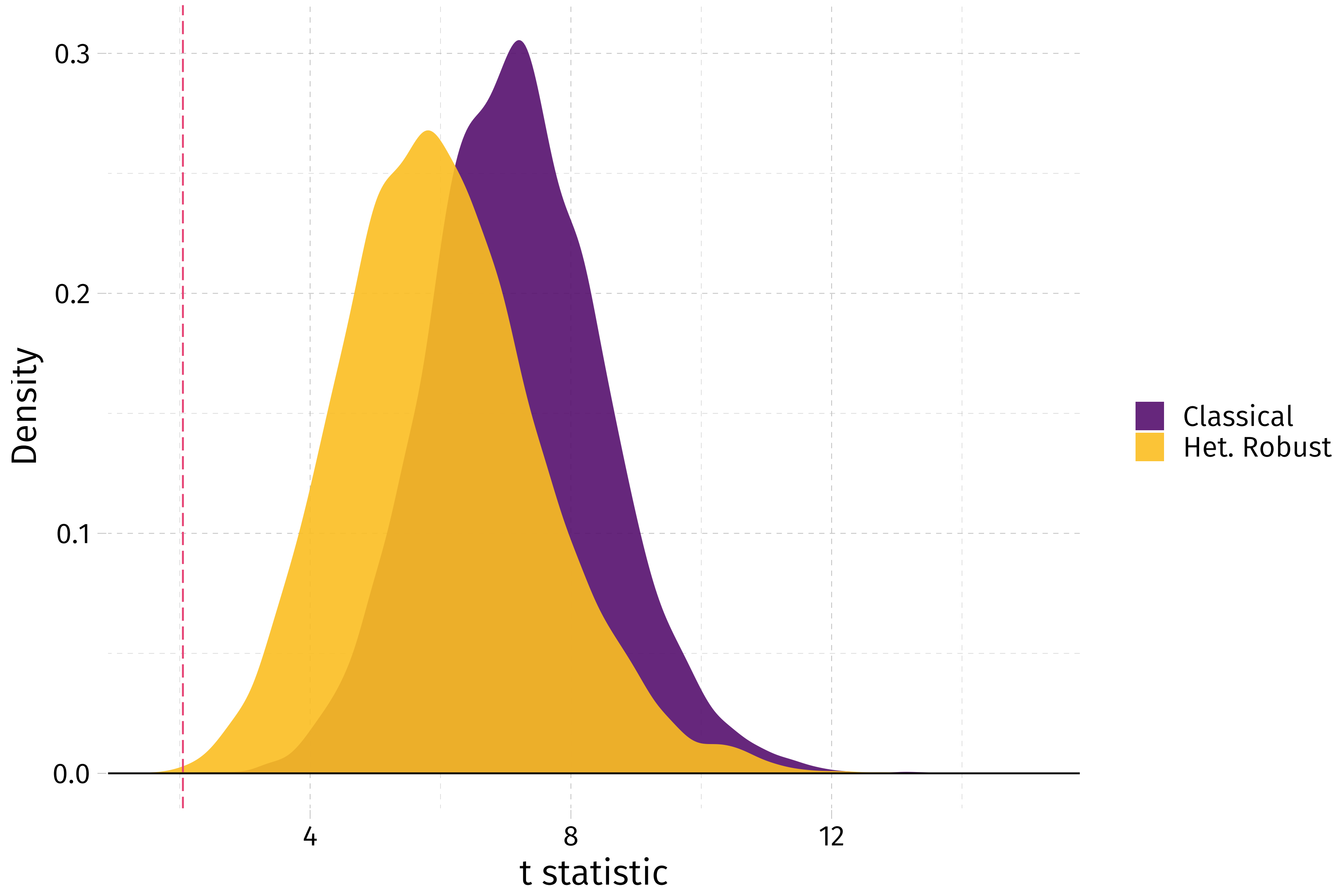
```r
# Bind list together
sim_df ← bind_rows(sim_list)
```

So what are the results?

Comparing the distributions of standard errors for the coefficient on x



Comparing the distributions of t statistics for the coefficient on x



Q All of these test are for a false H_0. How would the simulation change to enforce a *true* null hypothesis?

Simulation

Updating to enforce the null

Let's update our simulation function to take arguments `γ` and `δ` such that

$$\mathbf{Y}_i = 1 + e^{\gamma \mathbf{X}_i} + \varepsilon_i$$

where $\varepsilon_i \sim \mathrm{N}(0, \sigma^2 \mathbf{X}_i^\delta)$.

In other words,

- $\gamma = 0$ implies no relationship between $\mathbf{Y}_i$ and $\mathbf{X}_i$.
- $\delta = 0$ implies homoskedasticity.

Simulation

Updating to enforce the null

Updating the function…

```r
flex_iter ← function(iter, γ = 0, δ = 1, n = 30) {
  # Generate data
  iter_df ← tibble(
    x = runif(n, min = 0, max = 10),
    ε = rnorm(n, sd = 15 * x^δ),
    y = 1 + exp(γ * x) + ε
  )
  # Estimate models
  lm1 ← lm_robust(y ~ x, data = iter_df, se_type = "classical")
  lm2 ← lm_robust(y ~ x, data = iter_df, se_type = "HC2")
  # Stack and return results
  bind_rows(tidy(lm1), tidy(lm2)) %>%
    select(1:5) %>% filter(term == "x") %>%
    mutate(se_type = c("classical", "HC2"), i = iter)
}
```

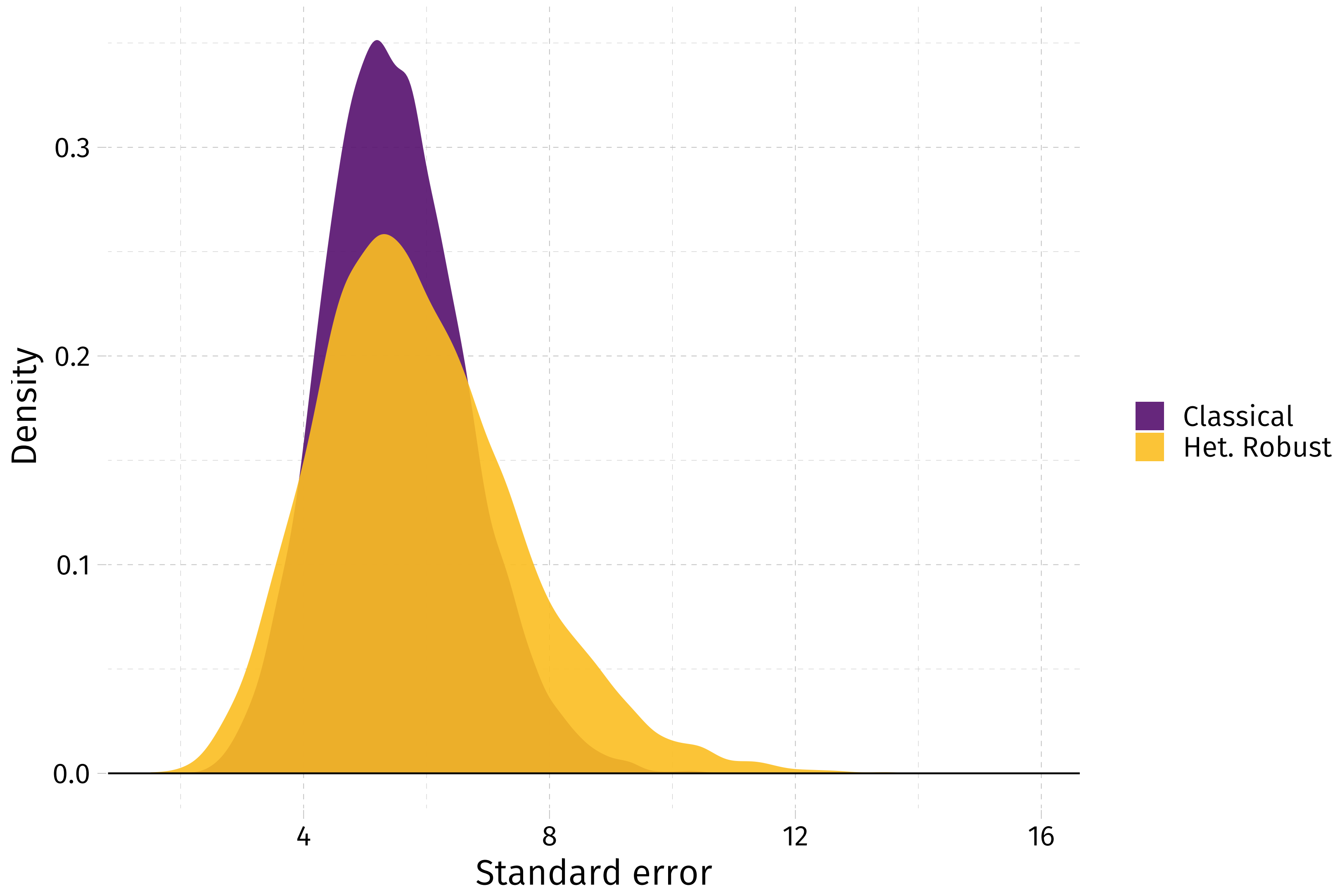
Simulation

Run again!

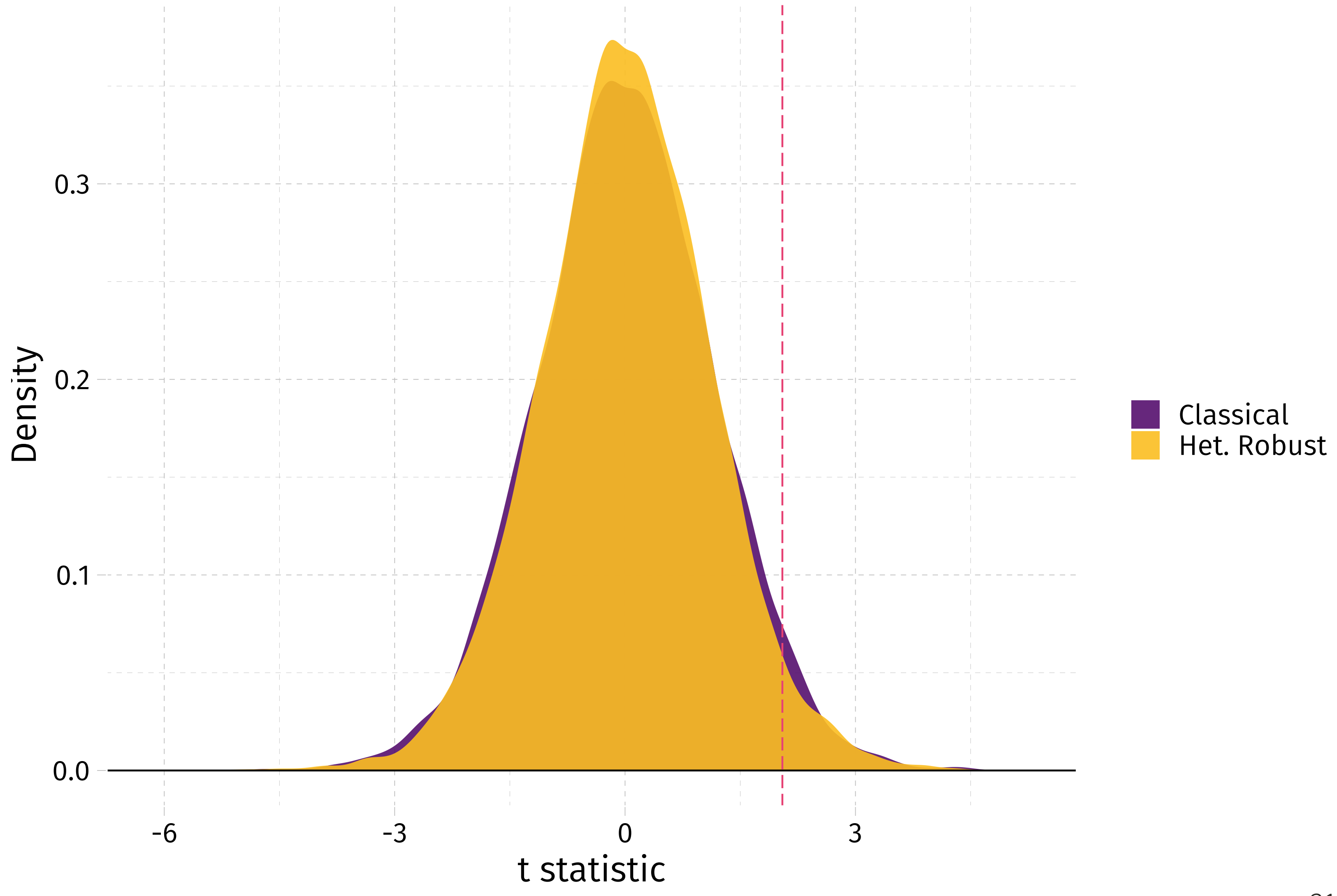
Now we run our new function `flex_iter()` 10,000 times

```r
# Packages
p_load(purrr, furrr)
# Set options
set.seed(123)
# Tell R to parallelize
plan(multiprocess)
# Run 10,000 iterations
null_df ← future_map(
  1:1e4, flex_iter,
  # Enforce the null hypothesis
  γ = 0,
  # Specify heteroskedasticity
  δ = 1,
  .options = future_options(seed = T)
) %>% bind_rows()
```

Comparing the distributions of standard errors for the coefficient on x



Comparing the distributions of t statistics for the coefficient on x



Distributions of *p*-values: both methods slightly over-reject the (true) null

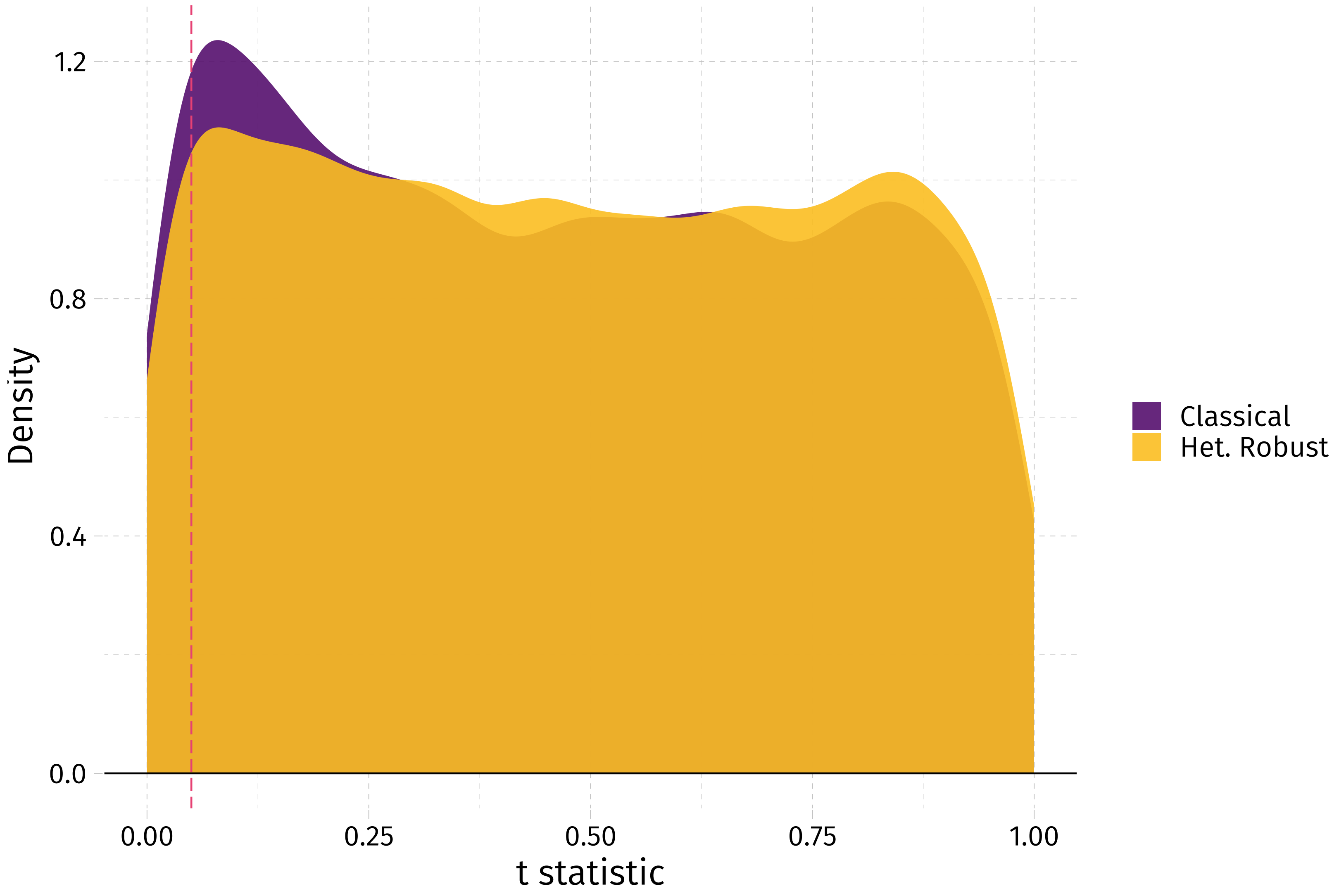


Table of contents

Admin

1. Schedule

Inference

1. Why?
2. OLS
3. Heteroskedasticity
4. Small-sample warning
5. Simulation
    - Outline
    - DGP
    - Iterating
    - Parallelization
    - Results
    - Under the null