Inference and Randomization

EC 425/525, Set 11

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05 June 2019

Prologue

Schedule

Last time

An analytical solution to cluter-robust inference

Today

Inference using (re)randomization [†]

Upcoming

The end is near. As is the final.

† These notes follow notes by Kosuke Imai, *Field Experiments* by Gerber and Green, and *Causal Inference for Statistics, Social, and Biomedical Sciences* by Imbens and Rubin.

Inference and (re)randomization

Inference recap

Our inference techniques have focused on (asymptotic) **analytical methods**.

1. Choose (or derive) an estimator

2. Derived the estimator's (asymptotic) distribution[†]

3. Construct confidence intervals or hypothesis tests

† And, consequently, standard errors.

Resampling

Resampling methods offers a different, more computationally intense (less asymptoically intense) approach.

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A **resampling method** involves repeatedly drawing samples (*resampling*) from a dataset and refitting the model of interest on each sample. We can learn about the behavior of the model through its performance across the many iterations.[†]

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Common implementations: Bootstrap (and jackknife), cross validation, permutation tests/randomization inference

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The bootstrap

The bootstrap

Basics

Bootstrapping resamples, *with replacement*, from the original dataset.

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Bootstrapping resamples, *with replacement*, from the original dataset.

- In each sample, we apply our estimator.
- Then, we consider the distribution/properties of these estimates.

This resampling helps us better understand the uncertainty associated with our estimator (within the current data setting).

The bootstrap

More formally

Let's formalize the bootstrap a bit.

- Z denotes our original dataset (*e.g.*, $Z = [\mathbf{Y} \mid \mathbf{X}]$ in our standard setup).

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- $\hat{\alpha}^{\star 1} = \hat{\alpha}(Z^{\star 1})$ is our estimator evaluated on the first bootstrap sample.

The **bootstrapped standard error** of $\hat{\alpha}$ is the standard deviation of the $\hat{\alpha}^{\star b}$

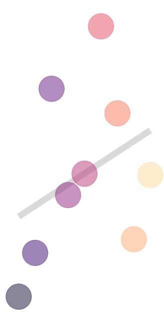
$$\mathrm{SE}_B(\hat{\alpha}) = \sqrt{\frac{1}{B} \sum_{b=1}^{B} \left(\hat{\alpha}^{\star b} - \frac{1}{B} \sum_{\ell=1}^{B} \hat{\alpha}^{\star \ell} \right)^2}$$

The bootstrap

More graphically

Z

$\hat{\beta} = 0.653$



More graphically

Z

7	8	9
4	5	6
1	2	3

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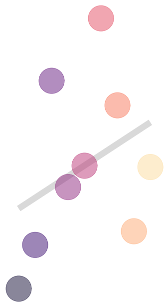
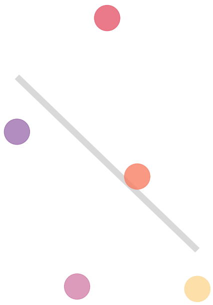


$Z^{\star 1}$

6	7	5
7	6	9
3	9	9

$\hat\beta = -0.961$



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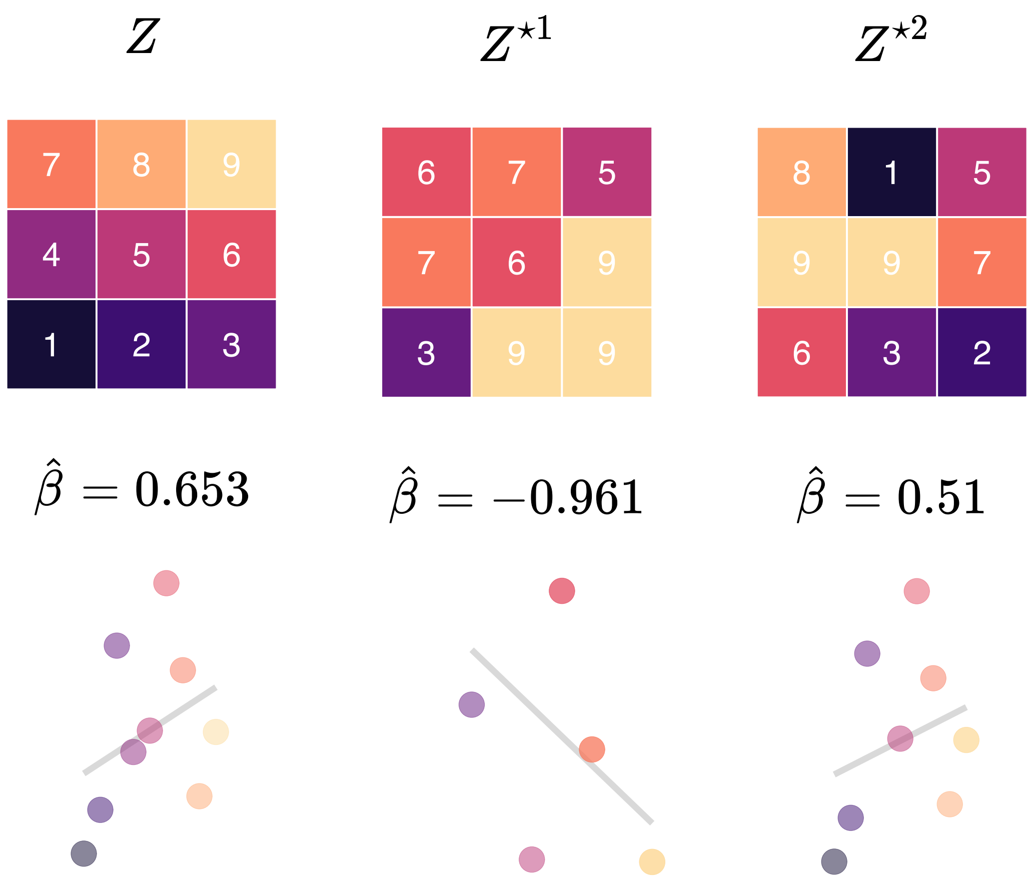

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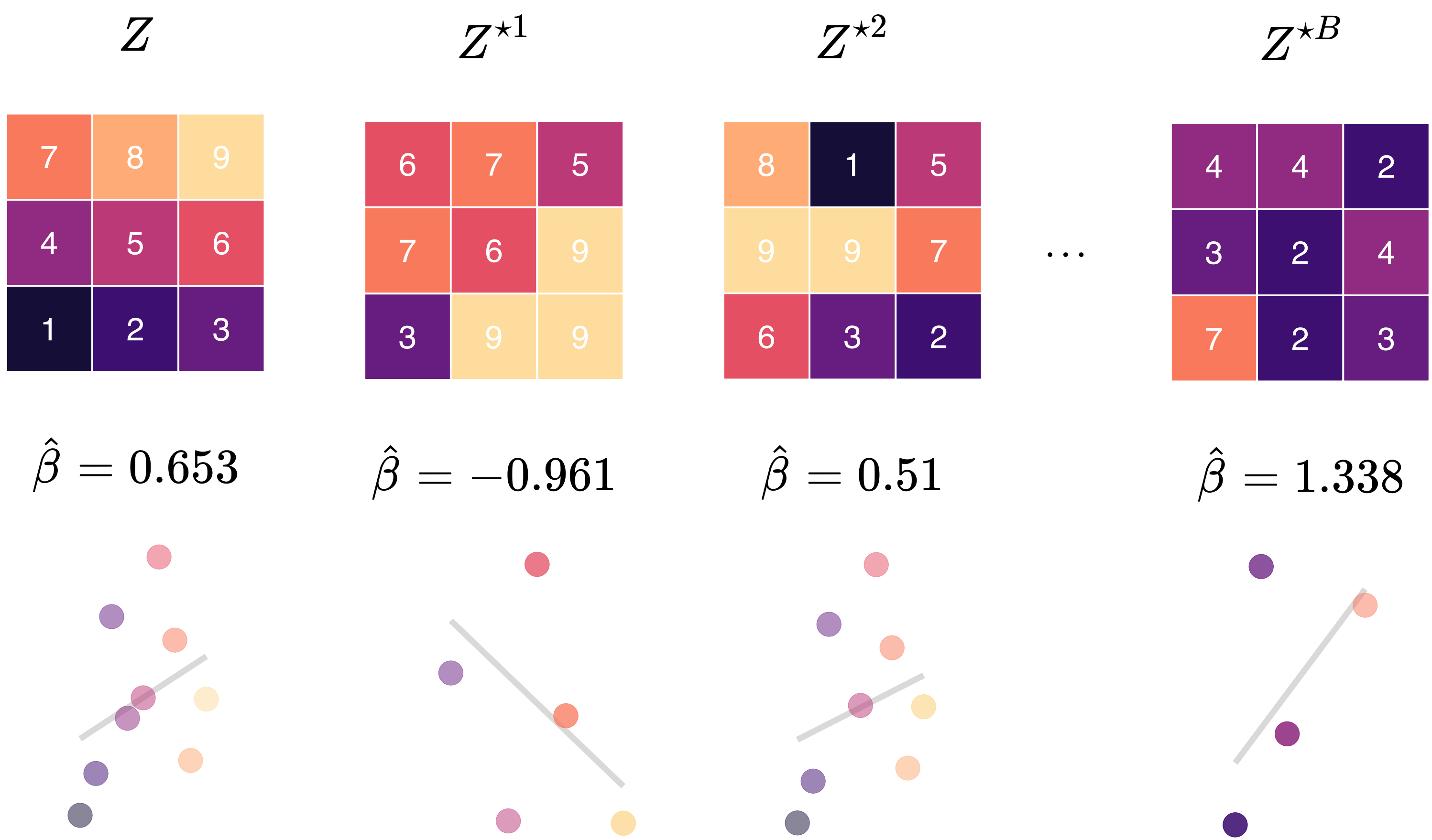
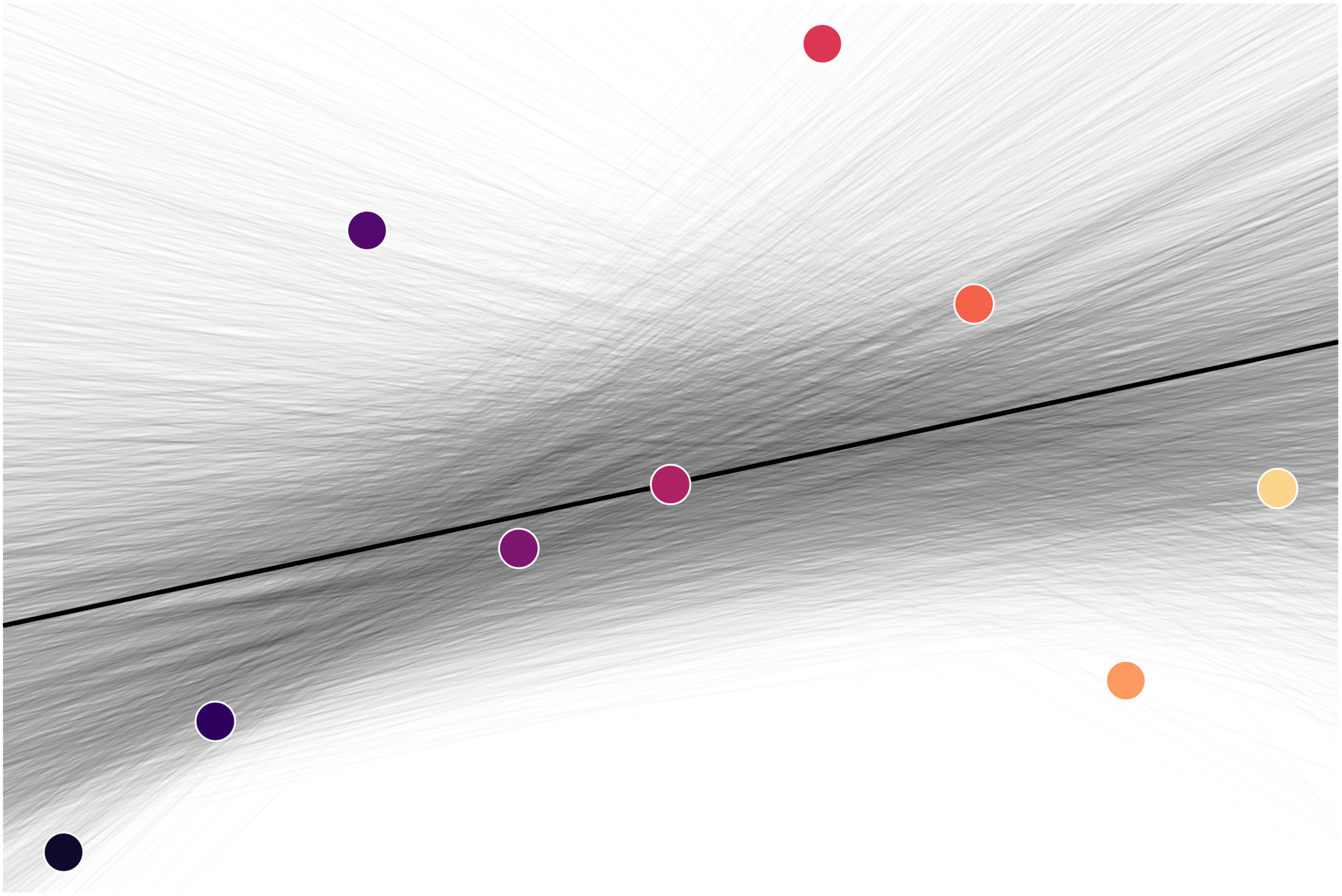


The bootstrap

Running this bootstrap 10,000 times

```
plan(multiprocess, workers = 10)
# Set a seed
set.seed(123)
# Run the simulation 1e4 times
boot_df <- future_map_dfr(
  # Repeat sample size 100 for 1e4 times
  rep(n, 1e4),
  # Our function
  function(n) {
    # Estimates via bootstrap
    est <- lm(y ~ x, data = z[sample(1:n, n, replace = T), ])
    # Return a tibble
    data.frame(int = est$coefficients[1], coef = est$coefficients[2])
  },
  # Let furrr know we want to set a seed
  .options = future_options(seed = T)
)
```



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If we go the old-fashioned OLS route $\left(s^2\left(\mathbf{X}'\mathbf{X}\right)^2\right)$, we estimate 0.673.

Not bad.

Permutation tests

Permutation tests

Motivation

Consider the null hypothesis of *no average treatment effect*, *i.e.*,

$$\mathrm{H}_0\colon \overline{\mathrm{Y}}_0 = \overline{\mathrm{Y}}_1 \quad \left(\implies \bar{\tau} = 0\right)$$

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We've discussed how randomization avoids the pitfalls of selection bias.

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Randomization can also clarify inference—helping quantify uncertainty.

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We use the **experimental design**, rather than a probability model.

Permutation tests

Tea drinkers

Classic example Sir R. A. Fisher had a colleague who claimed to be able to tell whether the tea was poured into milk *or* milk was poured into the tea.†

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If Fisher's colleague had no ability and simply guessed (H_0), what is the probability she would have guessed all 8 cups correctly?

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Fisher's H_0: the answers were unrelated to the cups' actual contents.

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This is the idea behind permutation testing and randomization inference.

Permutation tests

Tea drinkers with a vengeance

Cup
1
2
3
4
5
6
7
8

Permutation tests

Tea drinkers with a vengeance

Cup	*Guess*
1	**m**
2	**t**
3	**t**
4	**m**
5	**m**
6	**t**
7	**t**
8	**m**

Permutation tests

Tea drinkers with a vengeance

Cup	*Guess*	*Truth*
1	**m**	**m**
2	**t**	**t**
3	**t**	**t**
4	**m**	**m**
5	**m**	**m**
6	**t**	**t**
7	**t**	**t**
8	**m**	**m**
		8/8

Permutation tests

Tea drinkers with a vengeance

Cup	*Guess*	*Truth*	P_1
1	**m**	**m**	**m**
2	**t**	**t**	**~~m~~**
3	**t**	**t**	**~~m~~**
4	**m**	**m**	**m**
5	**m**	**m**	**~~t~~**
6	**t**	**t**	**t**
7	**t**	**t**	**t**
8	**m**	**m**	**~~t~~**
		8/8	4/8

Permutation tests

Tea drinkers with a vengeance

Cup	*Guess*	*Truth*	P_1	P_2
1	m	m	m	m
2	t	t	~~m~~	~~m~~
3	t	t	~~m~~	~~m~~
4	m	m	m	~~t~~
5	m	m	~~t~~	m
6	t	t	t	t
7	t	t	t	t
8	m	m	~~t~~	~~t~~
		8/8	4/8	4/8

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Cup	*Guess*	*Truth*	P_1	P_2	P_3
1	m	m	m	m	m
2	t	t	~~m~~	~~m~~	~~m~~
3	t	t	~~m~~	~~m~~	~~m~~
4	m	m	m	~~t~~	~~t~~
5	m	m	~~t~~	m	~~t~~
6	t	t	t	t	~~m~~
7	t	t	t	t	t
8	m	m	~~t~~	~~t~~	~~t~~
		8/8	4/8	4/8	2/8

Permutation tests

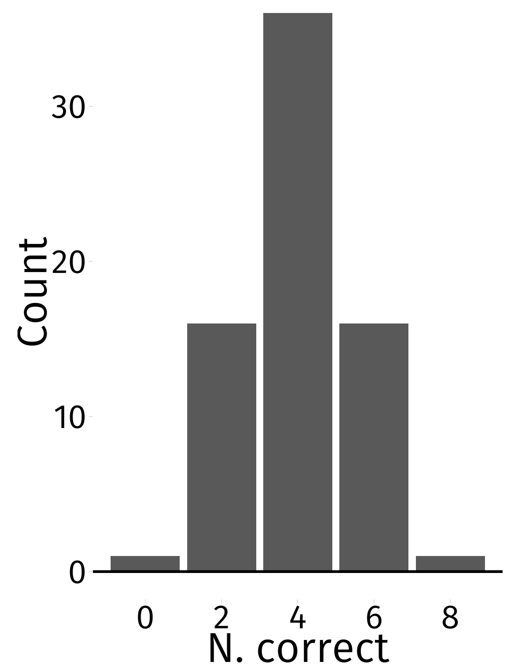
Tea drinkers with a vengeance

Cup	*Guess*	*Truth*	P_1	P_2	P_3	$\cdots$	P_{70}
1	m	m	m	m	m		~~t~~
2	t	t	~~m~~	~~m~~	~~m~~		t
3	t	t	~~m~~	~~m~~	~~m~~		t
4	m	m	m	~~t~~	~~t~~		~~t~~
5	m	m	~~t~~	m	~~t~~		m
6	t	t	t	t	~~m~~		~~m~~
7	t	t	t	t	t		~~m~~
8	m	m	~~t~~	~~t~~	~~t~~		m
		8/8	4/8	4/8	2/8		4/8

Permutation tests

Tea drinkers with a vengeance

Cup	*Guess*	*Truth*	P_1	P_2	P_3	$\cdots$	P_{70}
1	m	m	m	m	m		~~t~~
2	t	t	~~m~~	~~m~~	~~m~~		t
3	t	t	~~m~~	~~m~~	~~m~~		t
4	m	m	m	~~t~~	~~t~~		~~t~~
5	m	m	~~t~~	m	~~t~~		m
6	t	t	t	t	~~m~~		~~m~~
7	t	t	t	t	t		~~m~~
8	m	m	~~t~~	~~t~~	~~t~~		m
		8/8	4/8	4/8	2/8		4/8



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1	m	m	m	m	m		~~t~~
2	t	t	~~m~~	~~m~~	~~m~~		t
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4	m	m	m	~~t~~	~~t~~		~~t~~
5	m	m	~~t~~	m	~~t~~		m
6	t	t	t	t	~~m~~		~~m~~
7	t	t	t	t	t		~~m~~
8	m	m	~~t~~	~~t~~	~~t~~		m
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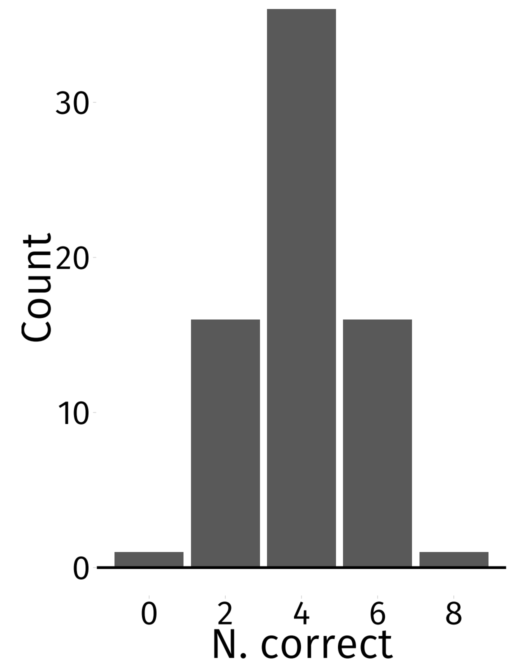

So our permutation-test-based *p*-value is $1/70 \approx 0.0143$. $\Longrightarrow$ Reject H_0.

Permutation tests

Generalization

The procedure for permutation-based hypothesis testing[†] is the same as our "standard" asymptotic-based hypothesis testing.

† Also called *Fisher's exact test*, as you get exact *p*-values.

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The procedure for permutation-based hypothesis testing[†] is the same as our "standard" asymptotic-based hypothesis testing.

1. **Define hypotheses**, H_o and H_a.
2. Choose our **rejection threshold** α (tolerated type-I error rate).
3. Choose a **test statistic** that is a function of our sample.
4. Derive/calculate the **test statistic's distribution *under H_o***.
5. **Compute the *p*-value** by comparing test stat. to its H_o distribution.
6. **Conclusions**—reject or fail to reject H_o.

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4. Derive/calculate the **test statistic's distribution *under* H_o**.
5. **Compute the *p*-value** by comparing test stat. to its H_o distribution.
6. **Conclusions**—reject or fail to reject H_o.

The difference: Permutation tests use the randomization's mechanism to construct the test-statistic's exact distribution under H_o.

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Permutation tests

More generally

Fisher focused on testing a **sharp null hypothesis**—no effect *for anyone*, *i.e.*,

$$\text{H}_\text{o}\text{: } \mathbf{Y}_{1i} - \mathbf{Y}_{0i} = \mathbf{0} \;\; \forall \boldsymbol{i} \quad \left(\implies \boldsymbol{\tau}_i = \mathbf{0} \;\; \forall \boldsymbol{i} \right)$$

against an alternative hypothesis that someone has a non-zero effect

$$\text{H}_\text{a}\text{: } \quad \mathbf{Y}_{1i} - \mathbf{Y}_{0i} \neq \mathbf{0} \text{ for some } \boldsymbol{i} \left(\implies \exists \boldsymbol{i} \;\; \textbf{s.t.} \;\; \boldsymbol{\tau}_i \neq \mathbf{0} \right)$$

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A **sharp null hypothesis** is specified *for all individuals*, *e.g.*,

$$\text{H}_\text{o}\text{: } \mathbf{Y}_{1i} - \mathbf{Y}_{0i} = \boldsymbol{C} \ \ \forall \boldsymbol{i}$$

which differs from the ATE-based nulls that we normally consider, *e.g.*,

$$\text{H}_\text{o}\text{: } \boldsymbol{E}[\mathbf{Y}_{1i} - \mathbf{Y}_{0i}] = \boldsymbol{C}.$$

Permutation tests

Key insight

Our estimate (or test statistic) is a function of

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- Permutations of $\mathbf{D}$ construct the *exact* null distribution (unchanged $\mathbf{Y}$).

The number of possible permutations can get big—*e.g.*, 500 treated and 500 control has 2.7×10^{299} options. Approximate the distribution by sampling.

Permutation tests

On average

The sharp null was central to Fisher's interpretation.

Neyman *et al.* (1935) extended[†] this idea of permutation-based tests to the average treatment effect (testing H_o: $E[\mathbf{Y}_{1i}] - E[\mathbf{Y}_{0i}] = \mathbf{0}$).

Neyman and others also added standard errors and confidence intervals.

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These extensions have come to be known as **randomization inference**.[††]

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Randomization inference

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Setup

In order to **generalize our null hypothesis to the average treatment effect**,

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we have to give up something.

1. If we want an exact null distribution, then we must **assume a uniform treatment effect**.

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2. If we want to avoid assuming $\tau_i = \overline{\tau} \;\; \forall i$, then we have to **accept a non-exact null distribution**. (We don't observe $\mathbf{Y}_{0i}$ for $\mathbf{D}_i = 1$.)

Randomization inference

Setup

In order to **generalize our null hypothesis to the average treatment effect**,

$$\text{H}_0\text{: } \overline{\tau} = 0 \implies E[\text{Y}_{1i} - \text{Y}_{0i}] = 0$$

we have to give up something.

1. If we want an exact null distribution, then we must **assume a uniform treatment effect**. (Assuming our way back to a sharp null.)

2. If we want to avoid assuming $\tau_i = \overline{\tau} \ \ \forall i$, then we have to **accept a non-exact null distribution**. (We don't observe $\mathbf{Y}_{0i}$ for $\mathbf{D}_i = 1$.)

If we don't like either option, then we need to go back to deriving asymptotic properties via probability modeling assumptions.

Implementation

Once we decide which simplification we're willing to accept, we proceed similarly to permutation tests:

- shuffle $\mathbf{D}$ in a way that mimics treatment assignment
- collect test statistics from each iteration

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- shuffle $\mathbf{D}$ in a way that mimics treatment assignment
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Note Monte Carlo simulations, bootstrap, permutation tests, and randomization all apply very similar processes.

(Which) Test statistics

We still need to choose a test statistic on which we base the *p*-value.

- The **actual estimate**—difference in means or coefficient
- **Transformed estimates**
- **Quantiles**, *e.g.*, the median
- ***t* statistic**
- **Rank** statistics

Randomization inference

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We still need to choose a test statistic on which we base the *p*-value.

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- ***t* statistic**
- **Rank** statistics

We can also extend this idea to **confidence intervals**.

E.g., Use the point estimates associated with the 2.5$^{\text{th}}$ and 95$^{\text{th}}$ percentiles to construct a 95% confidence interval.

Example

Back to the LaLonde NSW dataset. We previously estimated

- the NSW increased real earnings by $\hat{\beta}_1 \approx$ \$886.30
- (het.-robust) standard error of \$488.20
- *t* statistic $t_{\text{stat}} \approx$ 1.82 with *p*-value $\approx$ 0.0699

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Let's re-randomize treatment 10,000 times. In each **iteration** r, calculate

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2. t_{stat}^r, the ***t* statistic**

Then calculate the implied p-values using the location of $\hat{\beta}_1$ and t_{stat} in the distributions of $\hat{\beta}_1^r$ and t_{stat}^r, respectively.†

† Very similar exercise for confience intervals.

Randomization inference

Example: Re-randomization

The main decision is how to generate treatment.

Q Should we permute $\mathbf{D}$ or draw $\mathbf{D}_i$ for each individual?[†]

† The difference is in whether we hold the number of treated individuals constant.

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Randomization inference

Example: Re-randomization

The main decision is how to generate treatment.

Q Should we permute $\mathbf{D}$ or draw $\mathbf{D}_i$ for each individual?†

A How was the original randomization conducted?

We'll assume the NSW started with a set number of treatments to disperse.

† The difference is in whether we hold the number of treated individuals constant.

First, we'll write a function that performs one iteration.

```
# Arguments: 'i' (iteration), 'n_t' (# of trt)
fun_randomization ← function(i) {
  # Sample the treatment vector. NOTE: Sampling WITHOUT replacement
  t_i ← sample(nsw_df$treat, size = nrow(nsw_df), replace = F)
  # Regression using our re-randomized treatment
  est_i ← lm_robust(re78 ~ t_i, data = nsw_df) %>% tidy()
  # Return tibble with iteration, point estimate, and test statistic
  tibble(i, est = est_i[2,"estimate"], t_stat = est_i[2,"statistic"])
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}
```
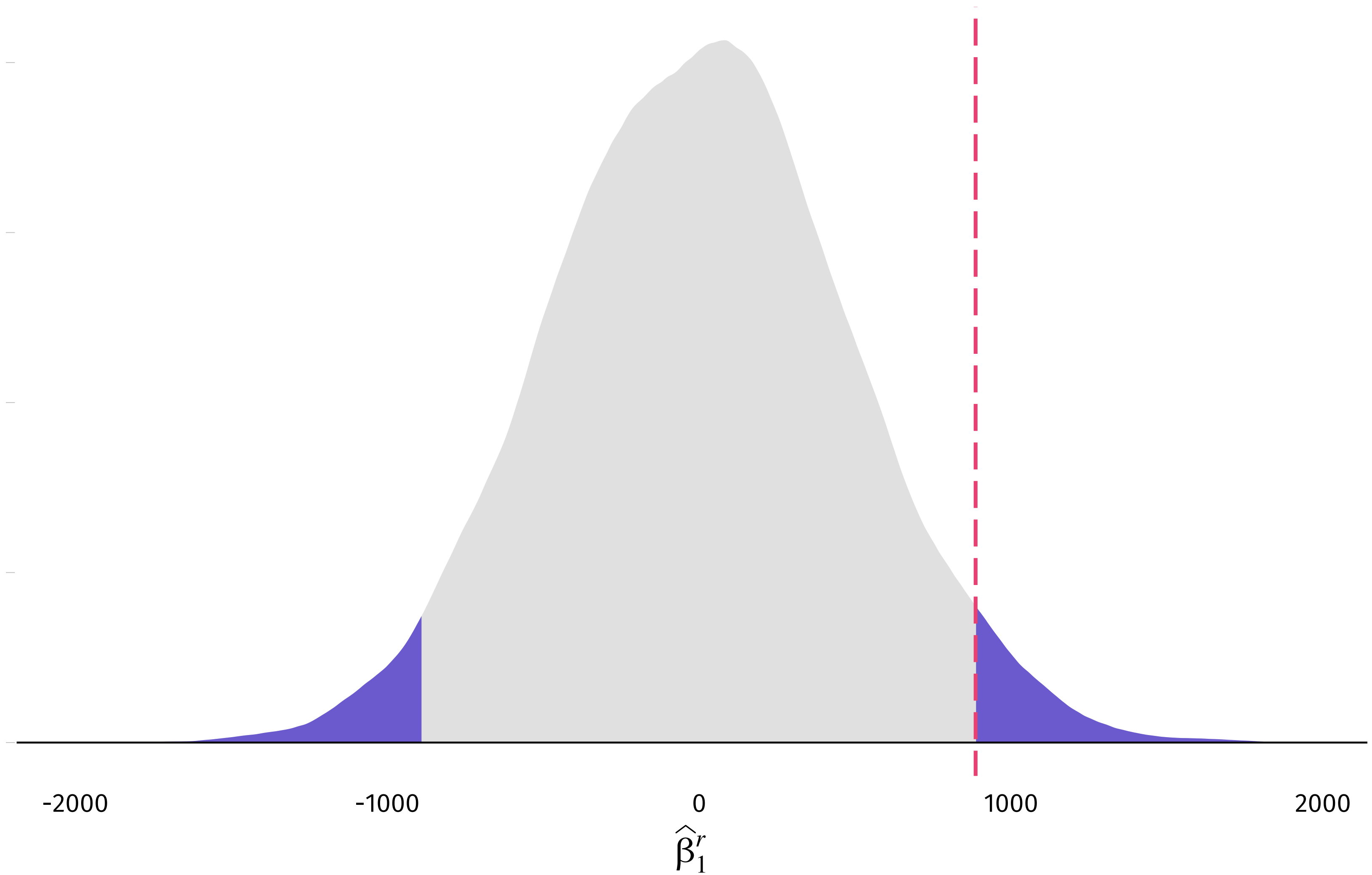
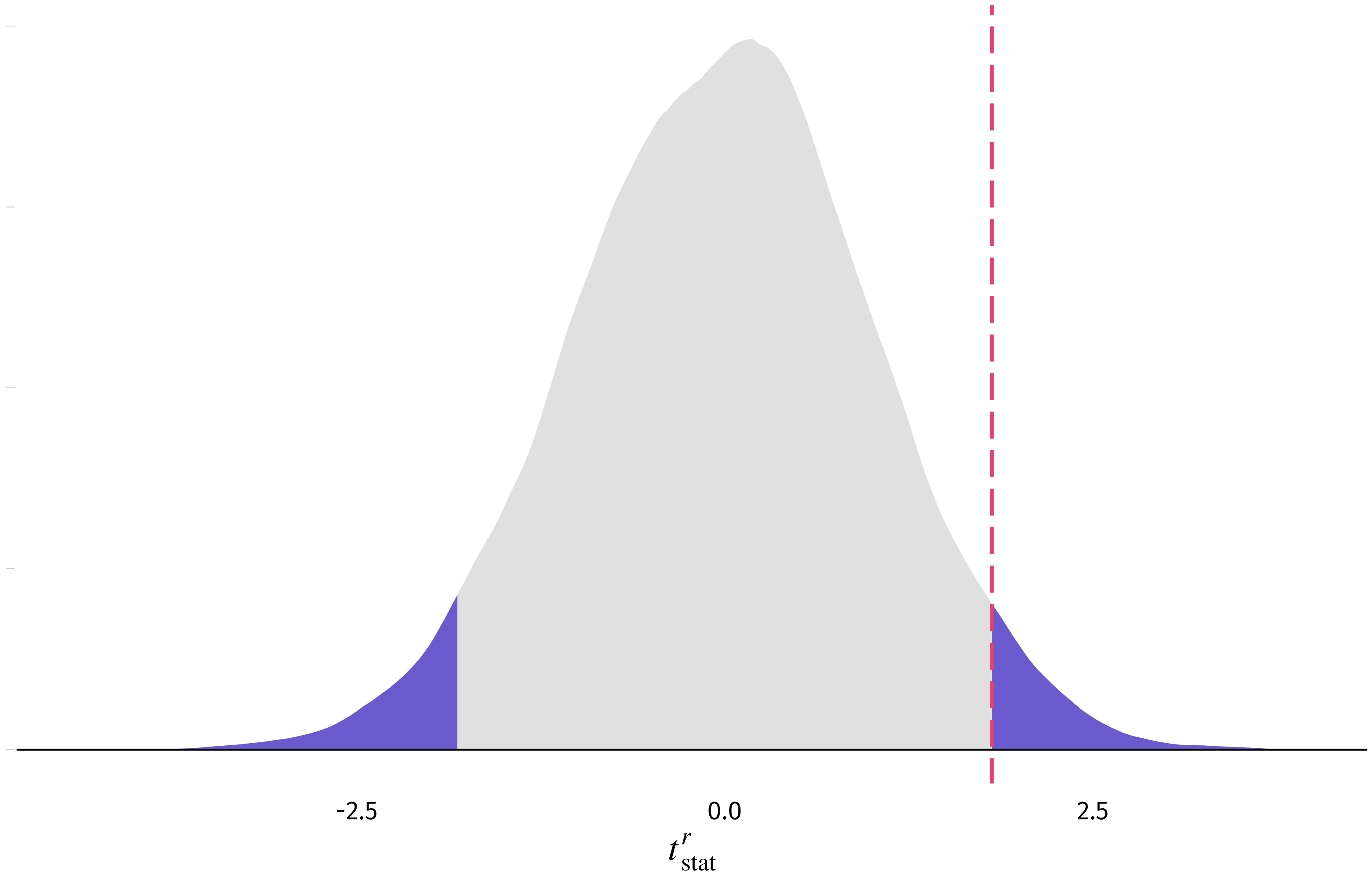

And now run the re-randomization function 10,000 times.

```r
# Set up parallelization and seed
plan(multiprocess, workers = 4); set.seed(1234)
# Run the simulation 1e4 times
random_df <- future_map_dfr(
  1:1e4,
  fun_randomization,
  .options = future_options(seed = T)
)
```

Result 1 Share $|\hat{\beta}_1^r| > \hat{\beta}_1 = 0.0624$. (Original p-value $= 0.0699$)



Result 2 Share $|t^r_{\text{stat}}| > t_{\text{stat}} = 0.0718$. (Original p-value = 0.0699)



Confidence intervals

To construct confidence intervals, we **invert** randomization-based **hypothesis tests**, imposing a range of null hypotheses.

Randomization inference

Confidence intervals

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E.g., To construct a 95% C.I. for $\hat{\tau}$

1. Impose the null hypothesis H_0: $\tau = \tau_o$ for many values of τ_o.
2. Find all values of τ_o that do not reject $\hat{\tau}$ at the 5% level.

Randomization inference

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Note We must to be able to clearly impose the null in our "model".

Athey and Imbens (2016) **on regression and randomization inference**:[†]

> Although these methods [regression] remain the most popular way of analyzing data from randomized experiments, we suggest caution in using them.
>
> … In particular there is a disconnect between the way the conventional assumptions in regression analyses are formulated and the implications of randomization. As a result it is easy for the researcher using regression methods to go beyond analyses that are justified by randomization, and end up with analyses that rely on a difficult-to-assess mix of randomization assumptions, modeling assumptions, and large sample approximation.

† Specifically in the context of experiments, though the concerns should remain in other contexts.

Athey and Imbens (2016) **on regression and randomization inference**:[†]

> Ultimately we recommend that researchers wishing to use regression or other model-based methods rather than the randomization-based methods we prefer, do so with care. For example, using only indicator variables based on partitioning the covariate space, rather than using multi-valued variables as covariates in the regression function preserves many of the finite sample properties that simple comparisons of means have, and leads to regression estimates with clear interpretations. In addition, in many cases the potential gains from regression adjustment can also be captured by careful ex ante design, that is, through stratified randomized experiments to be discussed in the next section, without the potential costs associated with ex post regression adjustment.

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Randomization and clustering

The plot thickens

Permutation tests and randomization inference both work because we know[†] the process through which treatment was randomly assigned.

† Or claim to understand.

The plot thickens

Permutation tests and randomization inference both work because we know† the process through which treatment was randomly assigned.

If treatment is correlated within groups, then our bootstraps, permutations, and re-randomizations need to reflect this dependence.

† Or claim to understand.

Further reading

Papers

Bootstrap-Based Improvements for Inference with Clustered Errors
Cameron, Gelbach, and Miller (2008)

Channeling Fisher: Randomization Tests and the Statistical Insignificance of Seemingly Significant Experimental Results
Young (2019)

The Econometrics of Randomized Experiments Athey and Imbens (2016)

Randomization Inference With Natural Experiments
Ho and Imai (2012)
Also: Notes by Kosuke Imai

Further reading

Books: Resampling methods and the bootstrap

An Introduction to Statistical Learning
James, Witten, Hastie, and Tibshirani

Elements of Statistical Learning
Hastie, Tibshirani, and Friedman

Books: Permutation tests and randomization inference

Causal Inference for Statistics, Social, and Biomedical Sciences
Imbens and Rubin

Field Experiments
Gerber and Green

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