Heteroskedasticity

EC 421, Set 04

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Prologue

R showcase

`R Markdown`

- Simple mark-up language for for combining/creating documents, equations, figures, **R**, and more
- Basics of `Markdown`
- *E.g.*, `**I'm bold**`, `*I'm italic*`, `I = "code"`

Econometrics with R

- (Currently) free, online textbook
- Written and published using **R** (and probably `R Markdown`)
- *Warning:* I haven't read the full book.

Related: Tyler Ransom has a great cheatsheet for econometrics.

Schedule

Last Time

We wrapped up our review.

Today

Heteroskedasticity

Schedule

This week

First assignment!

Submit **one file** (HTML, PDF, or Word) that includes

1. Your written answers
2. Figures and regression output
3. The **R** code that generated your answers.

This file should be a rendered RMarkdown or Quarto file (`.rmd` or `.qmd`).

Important

- We should be able to easily find your answers for each question.
- **Do not copy.** (You will receive a zero.)

Prerequisite material

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The χ^2 distribution

Some test statistics are distributed as χ^2 random variables.

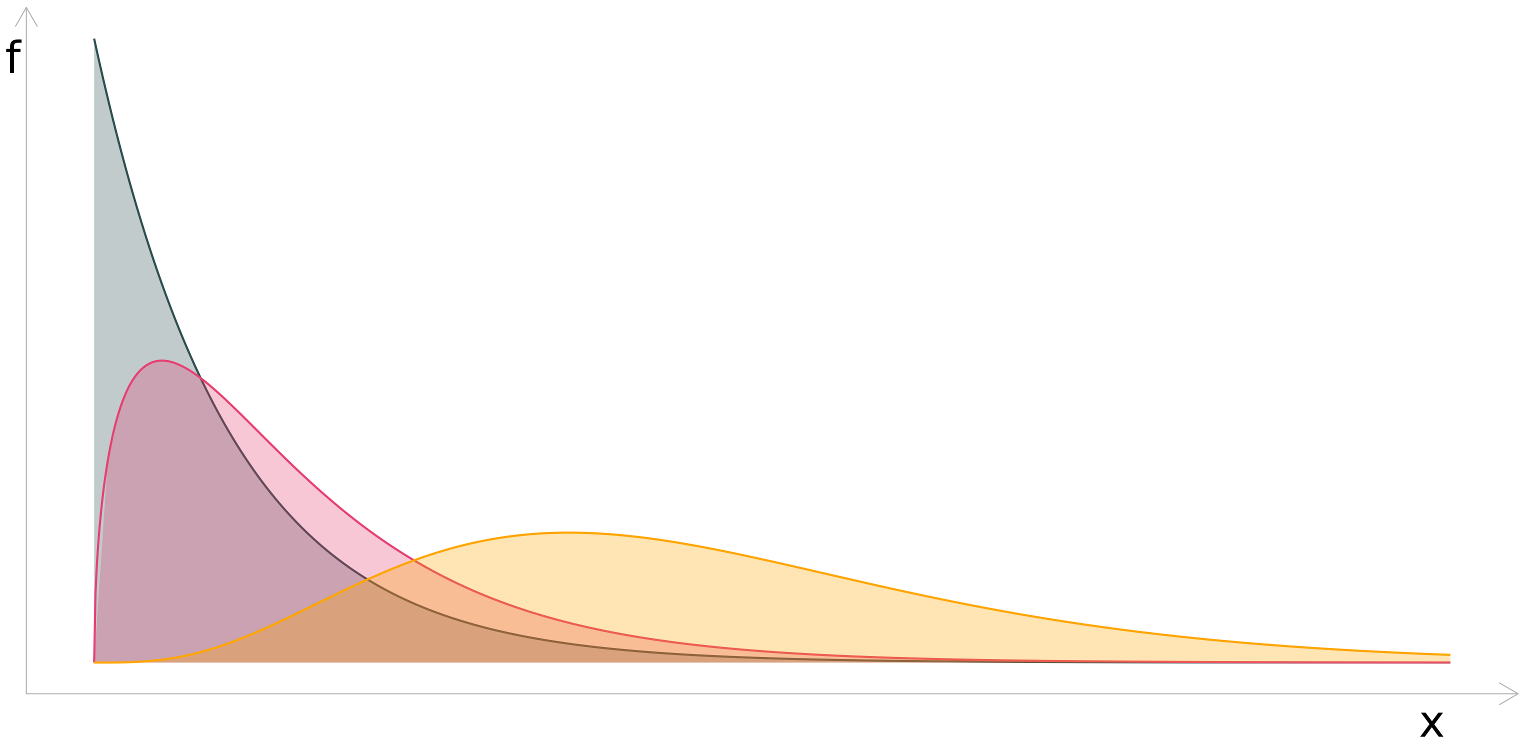
The χ^2 distribution is just another example of a common (named) distribution (like the Normal distribution, the t distribution, and the F).

The shape of the χ^2 distribution depends on a single parameter:

- We will call this parameter k
- Our test statistics will refer to k as *degrees of freedom*.

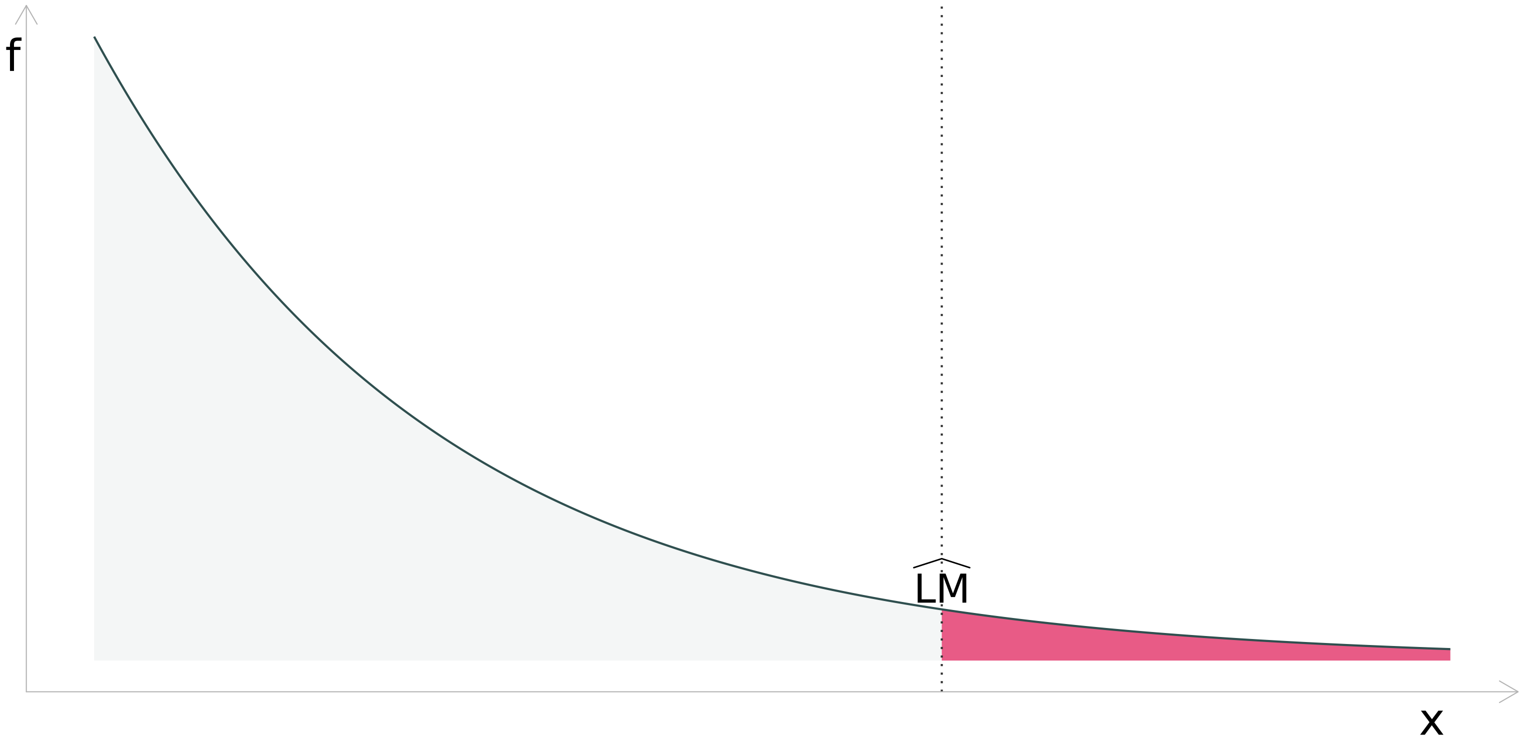
The χ^2 distribution

Three examples of χ^2_k: $k = 1$, $k = 2$, and $k = 9$



The χ^2 distribution

Probability of observing a more extreme test statistic $\widehat{\mathrm{LM}}$ under H_0



Heteroskedasticity

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5. The disurbances have **constant variance** σ^2 and **zero covariance**, *i.e.*,

   - $\boldsymbol{E}\left[u_i^2 \middle| X\right] = \mathrm{Var}(u_i|X) = \sigma^2 \implies \mathrm{Var}(u_i) = \sigma^2$
   - $\mathrm{Cov}\left(u_i, \, u_j \middle| X\right) = \boldsymbol{E}\left[u_i u_j \middle| X\right] = 0$ for $i \neq j$

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6. The disturbances come from a **Normal** distribution, *i.e.*, $u_i \overset{\text{iid}}{\sim} \text{N}\left(0, \sigma^2\right)$.

Heteroskedasticity

Today we're focusing on assumption #5:

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Heteroskedasticity: $\mathrm{Var}(u_i) = \sigma_i^2$ and $\sigma_i^2 \neq \sigma_j^2$ for some $i \neq j$.

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Violation of this assumption:

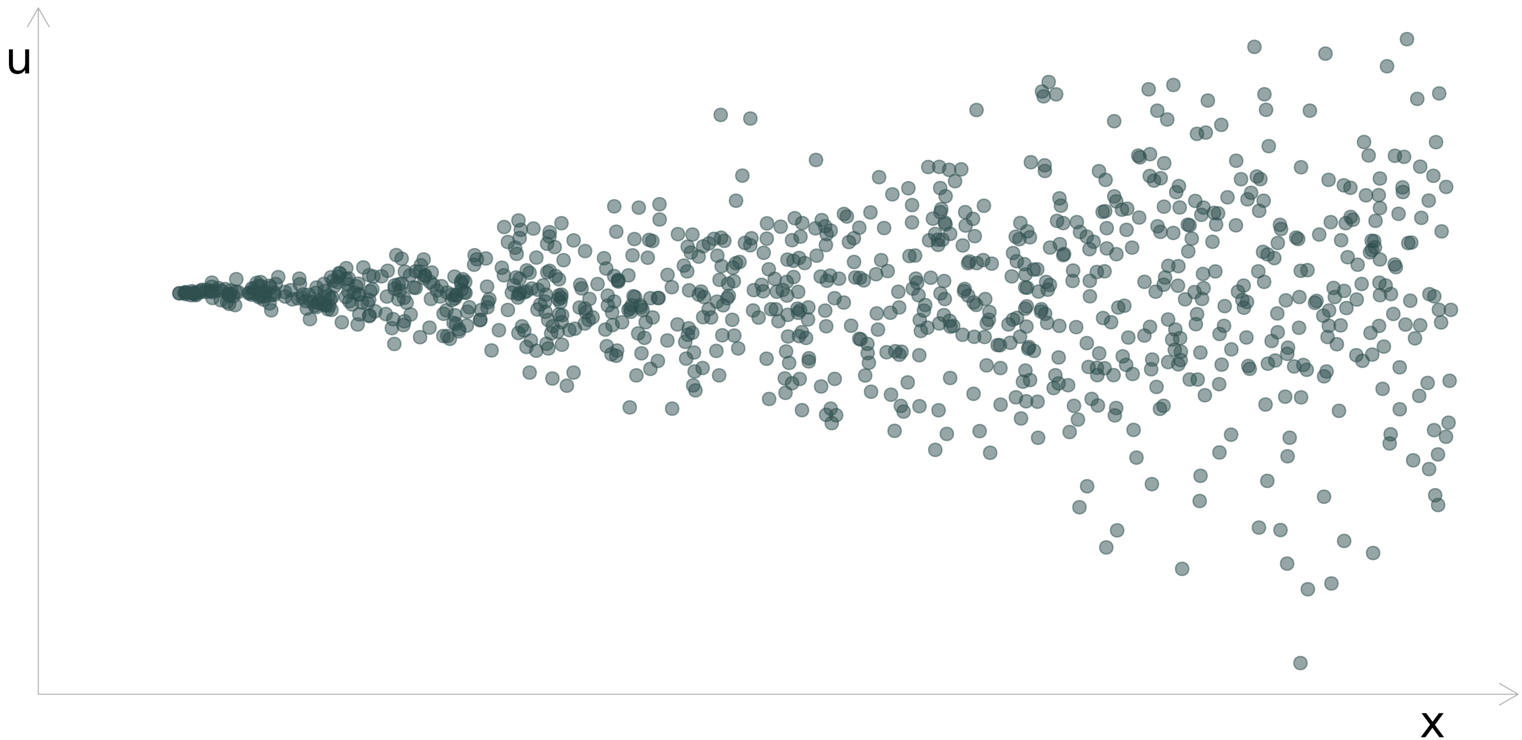
Heteroskedasticity: $\mathrm{Var}(u_i) = \sigma_i^2$ and $\sigma_i^2 \neq \sigma_j^2$ for some $i \neq j$.

In other words: Our disturbances have different variances.

Heteroskedasticity

Classic example of heteroskedasticity: The funnel

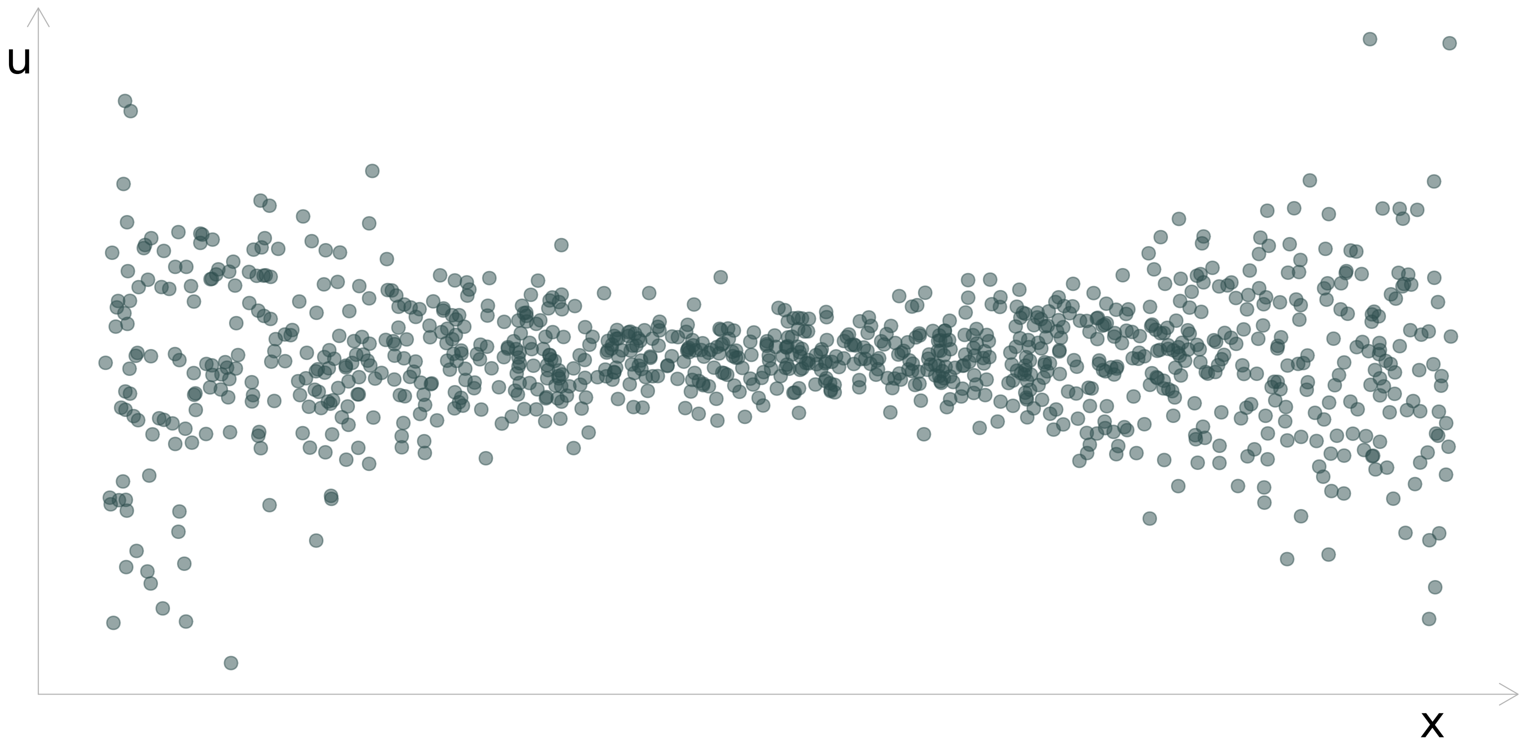
Variance of u increases with x



Heteroskedasticity

Another example of heteroskedasticity: (double funnel?)

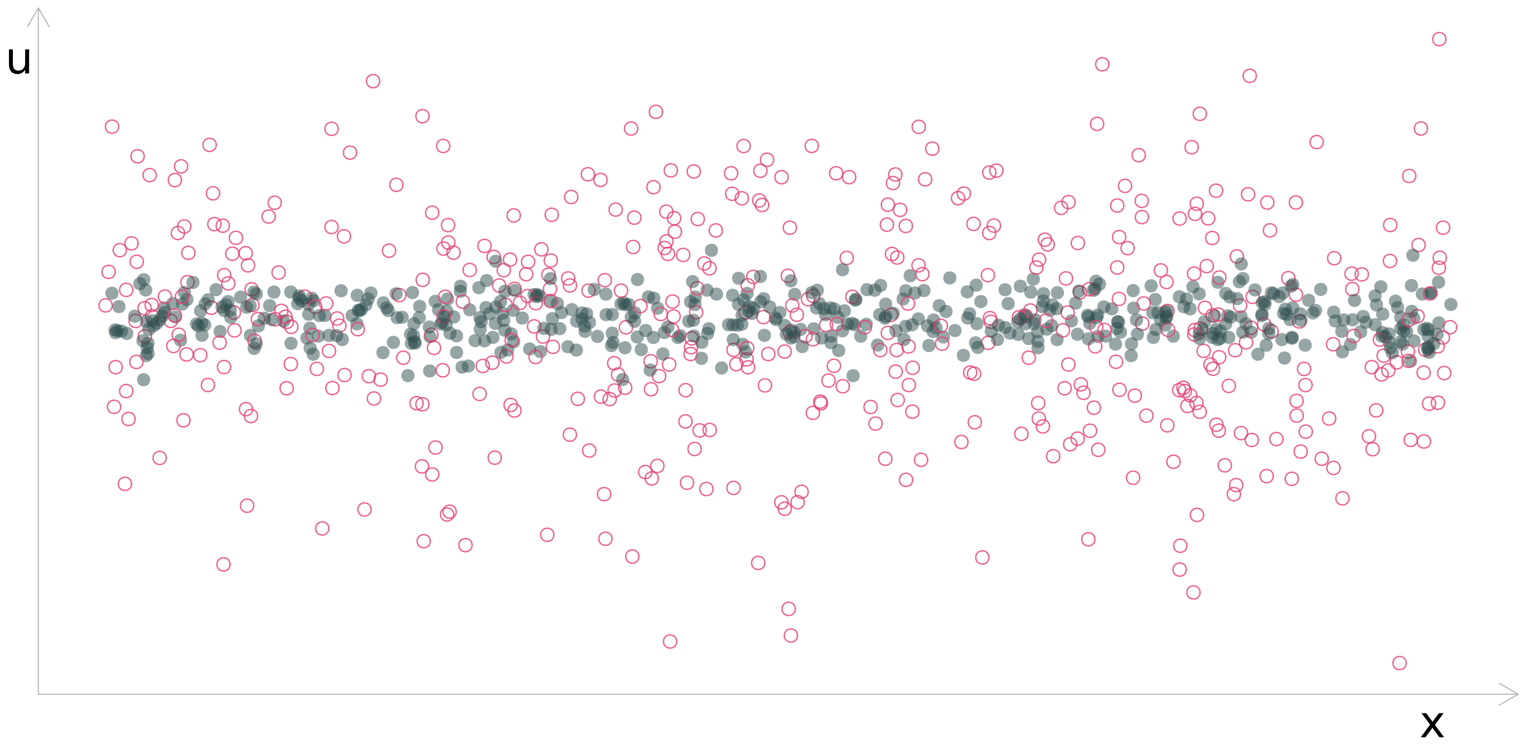
Variance of u increasing at the extremes of x



Heteroskedasticity

Another example of heteroskedasticity:

Differing variances of u by group



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Heteroskedasticity is present when the variance of u changes with any combination of our explanatory variables x_1, through x_k (henceforth: X).

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Why we care: Heteroskedasticity shows us how small violations of our assumptions can affect OLS's performance.

Heteroskedasticity

Consequences

So what are the consquences of heteroskedasticity? Bias? Inefficiency?

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Recall$_2$: We previously showed $\hat{\beta}_1 = \dfrac{\sum_i \left(y_i - \overline{y}\right)\left(x_i - \overline{x}\right)}{\sum_i \left(x_i - \overline{x}\right)^2}$

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It will actually help us to rewrite this estimator as

$$\hat{\beta}_1 = \beta_1 + \frac{\sum_i \left(x_i - \overline{x}\right) u_i}{\sum_i \left(x_i - \overline{x}\right)^2}$$

Heteroskedasticity

Proof: Assuming $y_i = \beta_0 + \beta_1 x_i + u_i$

$$
\begin{aligned}
\hat{\beta}_1 &= \frac{\sum_i \left(y_i - \overline{y}\right)\left(x_i - \overline{x}\right)}{\sum_i \left(x_i - \overline{x}\right)^2} \\
&= \frac{\sum_i \left(\left[\beta_0 + \beta_1 x_i + u_i\right] - \left[\beta_0 + \beta_1 \overline{x} + \overline{u}\right]\right)\left(x_i - \overline{x}\right)}{\sum_i \left(x_i - \overline{x}\right)^2} \\
&= \frac{\sum_i \left(\beta_1 \left[x_i - \overline{x}\right] + \left[u_i - \overline{u}\right]\right)\left(x_i - \overline{x}\right)}{\sum_i \left(x_i - \overline{x}\right)^2} \\
&= \frac{\sum_i \left(\beta_1 \left[x_i - \overline{x}\right]^2 + \left[x_i - \overline{x}\right]\left[u_i - \overline{u}\right]\right)}{\sum_i \left(x_i - \overline{x}\right)^2} \\
&= \beta_1 + \frac{\sum_i \left(x_i - \overline{x}\right)\left(u_i - \overline{u}\right)}{\sum_i \left(x_i - \overline{x}\right)^2}
\end{aligned}
$$

Heteroskedasticity

$$
\begin{aligned}
\hat{\beta}_1 = \cdots = \beta_1 &+ \frac{\sum_i \left(x_i - \overline{x}\right)\left(u_i - \overline{u}\right)}{\sum_i \left(x_i - \overline{x}\right)^2} \\
= \beta_1 &+ \frac{\sum_i \left(x_i - \overline{x}\right) u_i - \overline{u} \sum_i \left(x_i - \overline{x}\right)}{\sum_i \left(x_i - \overline{x}\right)^2} \\
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😅

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&= \beta_1 + \frac{\sum_i \left(x_i - \overline{x}\right)}{\sum_i \left(x_i - \overline{x}\right)^2} \underbrace{\boldsymbol{E}\left[u_i | X\right]}_{=0} \\
&= \beta_1
\end{aligned}$$

Phew. **OLS is still unbiased** for the β_k.

Consequences: Efficiency

OLS's **efficiency** and **inference** do not survive heteroskedasticity.

- In the presence of heteroskedasticity, OLS is **no longer the most efficient** (best) linear unbiased estimator.

Heteroskedasticity

Consequences: Efficiency

OLS's **efficiency** and **inference** do not survive heteroskedasticity.

- In the presence of heteroskedasticity, OLS is **no longer the most efficient** (best) linear unbiased estimator.

- It would be more informative (efficient) to **weight observations** inversely to their u_i's variance.

    - Downweight high-variance u_i's (too noisy to learn much).

    - Upweight observations with low-variance u_i's (more 'trustworthy').

    - Now you have the idea of weighted least squares (WLS)

Heteroskedasticity

Consequences: Inference

OLS **standard errors are biased** in the presence of heteroskedasticity.

- Wrong confidence intervals
- Problems for hypothesis testing (both t and F tests)

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Consequences: Inference

OLS **standard errors are biased** in the presence of heteroskedasticity.

- Wrong confidence intervals
- Problems for hypothesis testing (both t and F tests)
- It's hard to learn much without sound inference.

Heteroskedasticity

Solutions

1. **Tests** to determine whether heteroskedasticity is present.

2. **Remedies** for (1) efficiency and (2) inference

Testing for heteroskedasticity

Testing for heteroskedasticity

While we *might* have solutions for heteroskedasticity, the efficiency of our estimators depends upon whether or not heteroskedasticity is present.

1. The **Goldfeld-Quandt test**
2. The **White test**

Testing for heteroskedasticity

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1. The **Goldfeld-Quandt test**

2. The **White test**

Each of these tests[†] centers on the fact that we can **use the OLS residual e_i to estimate the population disturbance u_i**.

† There are many other options for testing, *e.g.*, the **Breusch-Pagan test**.

Testing for heteroskedasticity

The Goldfeld-Quandt test

Focuses on a specific type of heteroskedasticity: whether the variance of u_i differs **between two groups**.[†]

Remember how we used our residuals to estimate the σ^2?

$$s^2 = \frac{\text{SSE}}{n-1} = \frac{\sum_i e_i^2}{n-1}$$

We will use this same idea to determine whether there is evidence that our two groups differ in the variances of their disturbances, effectively comparing s_1^2 and s_2^2 from our two groups.

[†]: The G-Q test was one of the early tests of heteroskedasticity (1965).

Testing for heteroskedasticity

The Goldfeld-Quandt test

Operationally,

1. Order your the observations by x

2. Split the data into two groups of size n*

    - G_1: The first third
    - G_2: The last third

3. Run separate regressions of y on x for G_1 and G_2

4. Record SSE_1 and SSE_2

5. Calculate the G–Q test statistic

Testing for heteroskedasticity

The Goldfeld-Quandt test

The G-Q test statistic

$$F_{(n^\star-k,\, n^\star-k)} = \frac{\text{SSE}_2/(n^\star-k)}{\text{SSE}_1/(n^\star-k)} = \frac{\text{SSE}_2}{\text{SSE}_1}$$

follows an F distribution (under the null hypothesis) with $n^\star - k$ and $n^\star - k$ degrees of freedom.[†]

Testing for heteroskedasticity

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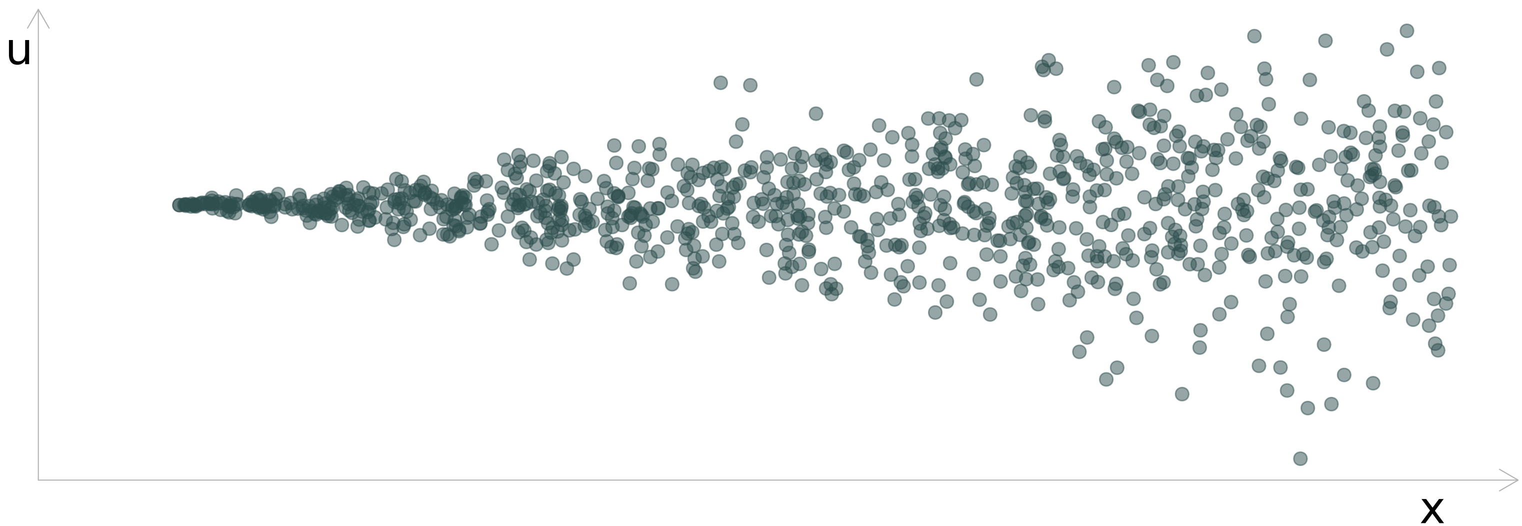
Notes

- The G-Q test requires the disturbances follow normal distributions.
- The G-Q assumes a very specific type/form of heteroskedasticity.
- Performs very well if we know the form of potentially heteroskedasticity.

[†]: Goldfeld and Quandt suggested $n^\star$ of $(3/8)n$. k gives number of estimated parameters (*i.e.*, $\hat{\beta}_j$'s).

Testing for heteroskedasticity

The Goldfeld-Quandt test



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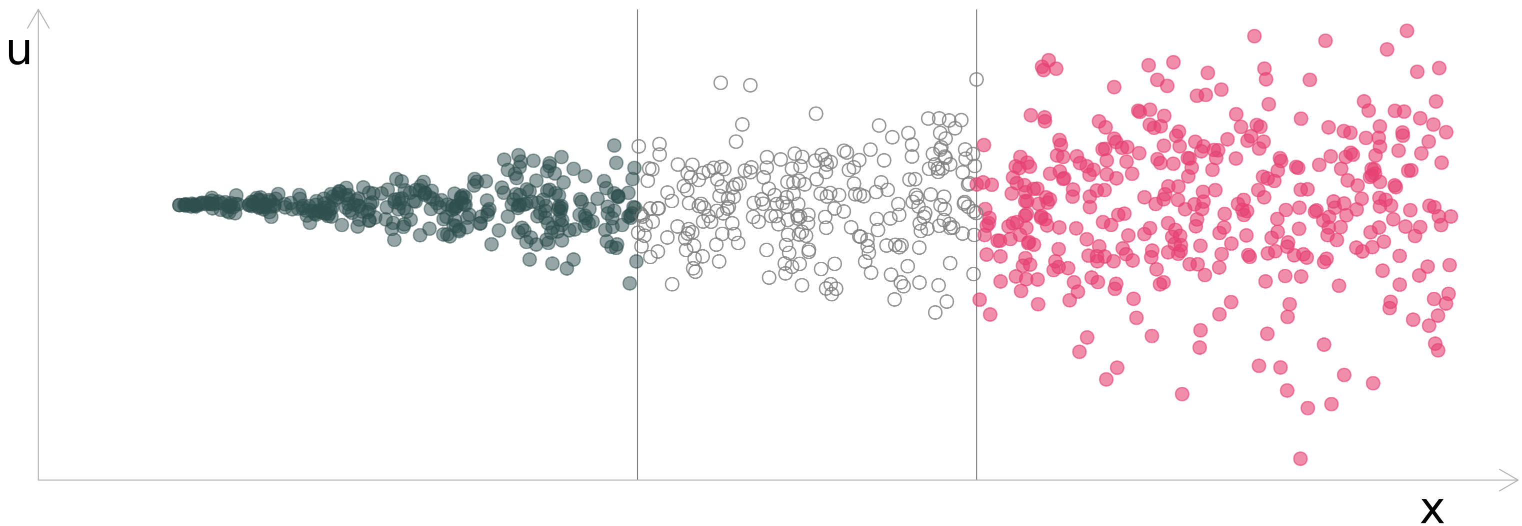


$$F_{375,\,375} = \frac{\text{SSE}_2 = 18,203.4}{\text{SSE}_1 = 1,039.5} \approx 17.5 \implies p\text{-value} < 0.001$$

$\therefore$ We reject H_0: $\sigma_1^2 = \sigma_2^2$ and conclude there is statistically significant evidence of heteroskedasticity.

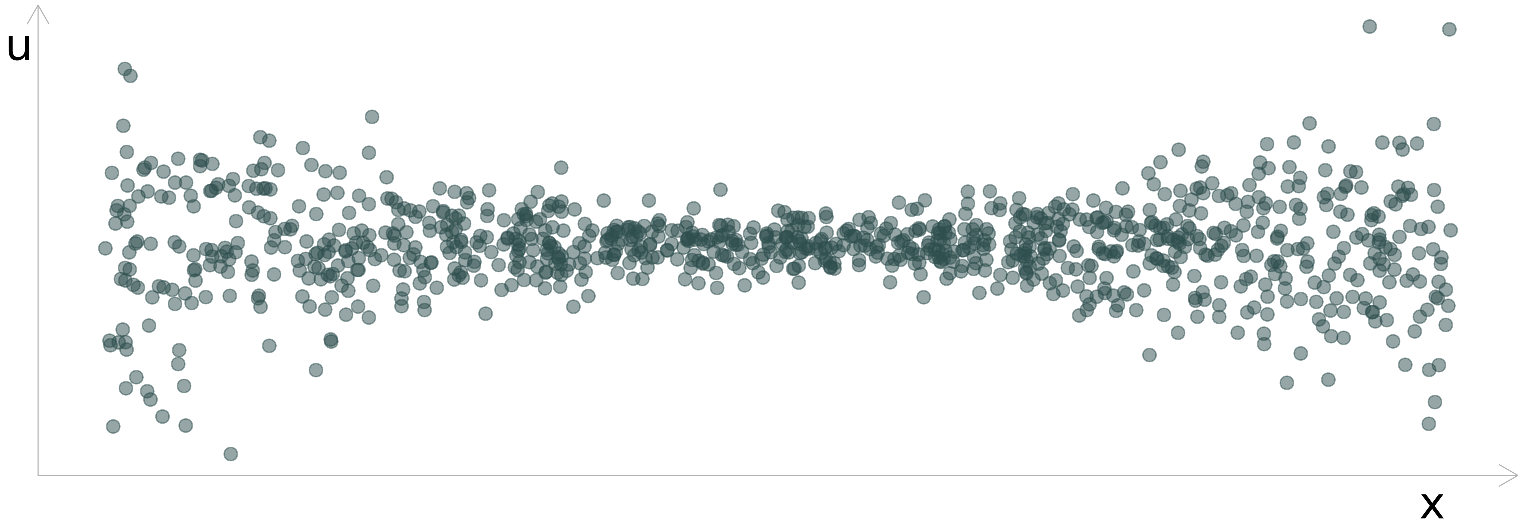
Testing for heteroskedasticity

The Goldfeld-Quandt test

The problem...

Testing for heteroskedasticity

The Goldfeld-Quandt test



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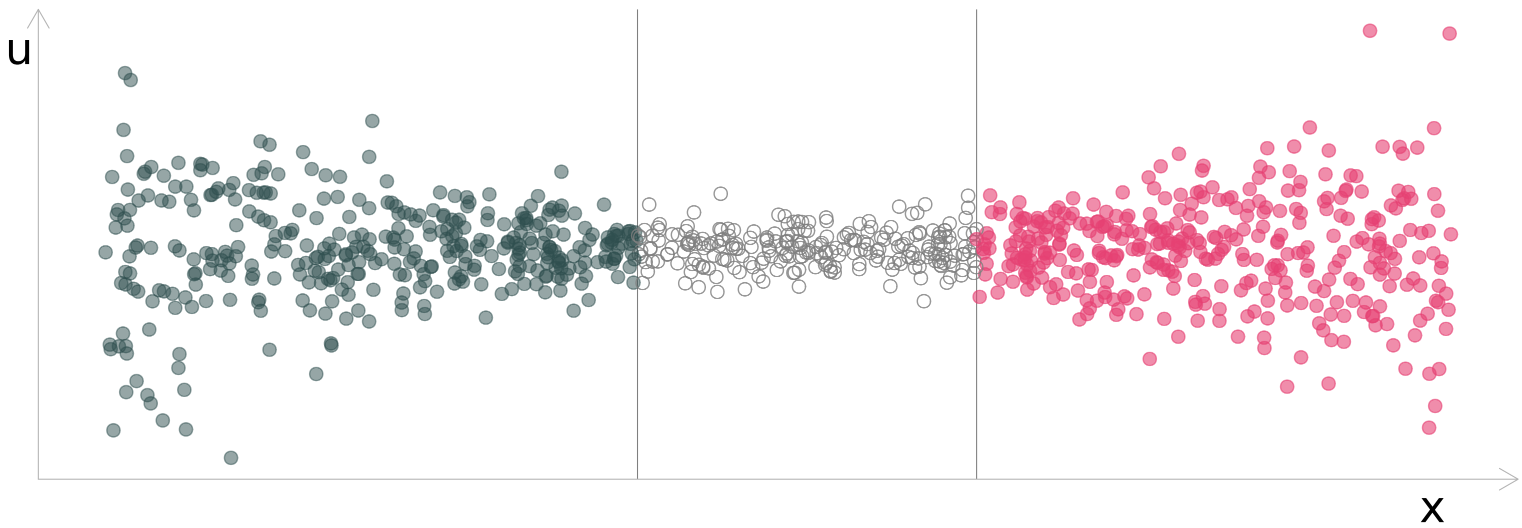

$$F_{375,\,375} = \frac{\text{SSE}_2 = 14,516.8}{\text{SSE}_1 = 14,937.1} \approx 1 \implies p\text{-value} \approx 0.609$$

$\therefore$ We fail to reject H_0: $\sigma_1^2 = \sigma_2^2$ while heteroskedasticity is present.

The White test

Breusch and Pagan (1981) attempted to solve this issue of being too specific with the functional form of the heteroskedasticity.

- Regress e_i^2 on $\boldsymbol{X} = [\mathbf{1}, \boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_k]$ and test for joint significance.
- Allows the data to show if/how the variance of u_i correlates with $\boldsymbol{X}$.
- If σ_i^2 correlates with $\boldsymbol{X}$, then we have heteroskedasticity.

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However, we actually want to know if

$$\sigma_1^2 = \sigma_2^2 = \cdots = \sigma_n^2$$

Q: Can't we just test this hypothesis?

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- Allows the data to show if/how the variance of $\boldsymbol{u}_i$ correlates with $\boldsymbol{X}$.
- If $\boldsymbol{\sigma}_i^2$ correlates with $\boldsymbol{X}$, then we have heteroskedasticity.

However, we actually want to know if

$$\boldsymbol{\sigma}_1^2 = \boldsymbol{\sigma}_2^2 = \cdots = \boldsymbol{\sigma}_n^2$$

Q: Can't we just test this hypothesis? **A:** Sort of.

The White test

Toward this goal, Hal White took advantage of the fact that we can **replace the homoskedasticity requirement with a weaker assumption**:

- **Old:** $\mathrm{Var}(u_i|X) = \sigma^2$
- **New:** u^2 is *uncorrelated* with the explanatory variables (*i.e.*, x_j for all j), their squares (*i.e.*, x_j^2), and the first-degree interactions (*i.e.*, $x_j x_h$).

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This new assumption is easier to explicitly test (*hint:* regression).

The White test

An outline of White's test for heteroskedasticity:

1. Regress y on x_1, x_2, …, x_k. Save residuals e.

2. Regress squared residuals on all explanatory variables, their squares, and interactions.

$$e^2 = \alpha_0 + \sum_{h=1}^{k} \alpha_h x_h + \sum_{j=1}^{k} \alpha_{k+j} x_j^2 + \sum_{\ell=1}^{k-1} \sum_{m=\ell+1}^{k} \alpha_{\ell,m} x_\ell x_m + v_i$$

3. Record R_e^2.

4. Calculate test statistic to test H_0: $\alpha_p = 0$ for all $p \neq 0$.

Testing for heteroskedasticity

The White test

White's test statistic is

$$\text{LM} = n \times R_e^2 \qquad \text{Under H}_0,\ \text{LM} \overset{\text{d}}{\sim} \chi_k^2$$

where R_e^2 comes from the regression of e^2 on the explanatory variables, their squares, and their interactions.

$$e^2 = \alpha_0 + \underbrace{\sum_{h=1}^{k} \alpha_h x_h}_{\text{Expl. variables}} + \underbrace{\sum_{j=1}^{k} \alpha_{k+j} x_j^2}_{\text{Squared terms}} + \underbrace{\sum_{\ell=1}^{k-1} \sum_{m=\ell+1}^{k} \alpha_{\ell,m} x_\ell x_m}_{\text{Interactions}} + v_i$$

Note: The k (for our χ_k^2) equals the number of estimated parameters in the regression above (the α_j), excluding the intercept (α_0).

Testing for heteroskedasticity

The White test

Practical note: If a variable is equal to its square (*e.g.*, binary variables), then you don't (can't) include it. The same rule applies for interactions.

Testing for heteroskedasticity

The White test

Example: Consider the model[†] $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + u$

Step 1: Estimate the model; obtain residuals (e).

Step 2: Regress e^2 on explanatory variables, squares, and interactions.

$$\begin{aligned} e^2 =& \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_1^2 + \alpha_5 x_2^2 + \alpha_6 x_3^2 \\ &+ \alpha_7 x_1 x_2 + \alpha_8 x_1 x_3 + \alpha_9 x_2 x_3 + v \end{aligned}$$

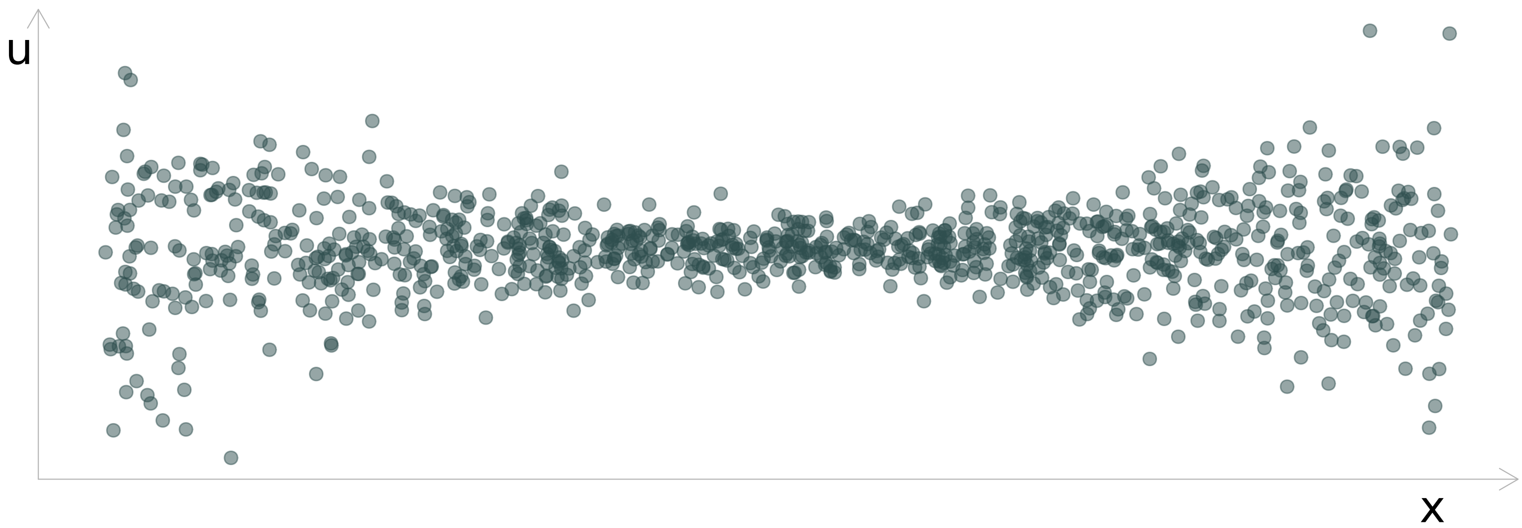
Record the R^2 from this equation (call it R_e^2).

Step 3: Test H_0: $\alpha_1 = \alpha_2 = \cdots = \alpha_9 = 0$ using $\text{LM} = n R_e^2 \overset{\text{d}}{\sim} \chi_9^2$.

[†]: To simplify notation here, I'm dropping the i subscripts.

Testing for heteroskedasticity

The White test



The White test for this simple linear regression.

$$e_i^2 = \hat{\alpha}_0 + \hat{\alpha}_1 x_{1i} + \hat{\alpha}_2 x_{1i}^2 \qquad \widehat{\text{LM}} = 185.8 \qquad p\text{-value} < 0.001$$

Testing for Heteroskedasticity

Examples

Testing for heteroskedasticity

Examples

Goal: Estimate the relationship between standardized test scores (outcome variable) and (1) student-teacher ratio and (2) income, *i.e.*,

$$\left(\textbf{Test score}\right)_i = \beta_0 + \beta_1 \textbf{Ratio}_i + \beta_2 \textbf{Income}_i + u_i \tag{1}$$

Potential issue: Heteroskedasticity... and we do not observe u_i.

Solution:

1. Estimate the relationship in (1) using OLS.
2. Test for heteroskedasticity.
   - Goldfeld-Quandt
   - White

Testing for heteroskedasticity

Examples

We will use testing data from the dataset `Caschool` in the `Ecdat` R package.

```r
# Load packages
library(pacman)
p_load(tidyverse, Ecdat)
# Select and rename desired variables; assign to new dataset
test_df = select(Caschool, test_score = testscr, ratio = str, income = avginc)
# Format as tibble
test_df = as_tibble(test_df)
# View first 2 rows of the dataset
head(test_df, 2)
```

```
#> # A tibble: 2 × 3
#>   test_score ratio income
#>        <dbl> <dbl>  <dbl>
#> 1       691.  17.9  22.7
#> 2       661.  21.5   9.82
```

Testing for heteroskedasticity

Examples

Let's begin by estimating our model

$$(\textbf{Test score})_i = \beta_0 + \beta_1 \textbf{Ratio}_i + \beta_2 \textbf{Income}_i + u_i$$

```
# Estimate the model
est_model = lm(test_score ~ ratio + income, data = test_df)
# Summary of the estimate
tidy(est_model)
```

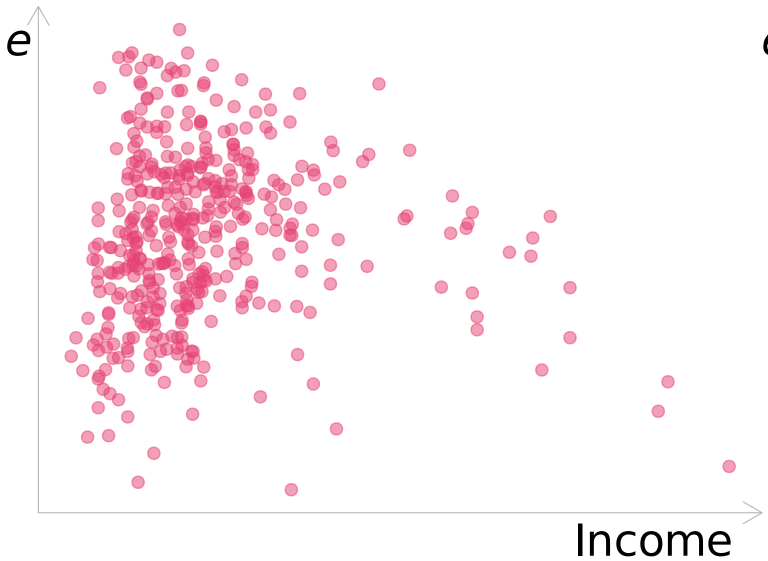
```
#> # A tibble: 3 × 5
#>   term        estimate std.error statistic   p.value
#>   <chr>          <dbl>     <dbl>     <dbl>     <dbl>
#> 1 (Intercept)  639.        7.45       85.7  5.70e-267
#> 2 ratio         -0.649    0.354      -1.83 6.79e-  2
#> 3 income         1.84     0.0928     19.8  4.38e- 62
```

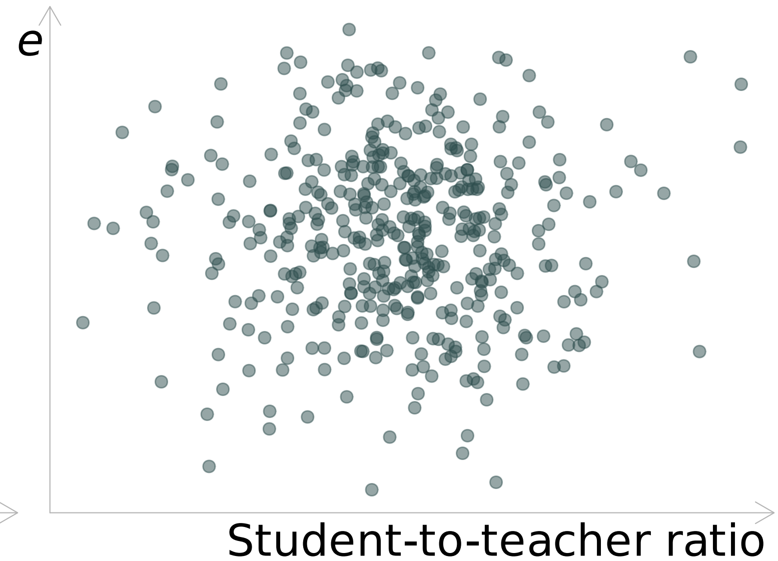
Testing for heteroskedasticity

Examples

Now, let's see what the residuals suggest about heteroskedasticity

```
# Add the residuals to our dataset
test_df$e = residuals(est_model)
```





Testing for heteroskedasticity

Example: Goldfeld-Quandt

Income looks potentially heteroskedastic; let's test via Goldfeld-Quandt.

```
# Arrange the data by income
test_df = arrange(test_df, income)
```

Testing for heteroskedasticity

Example: Goldfeld-Quandt

Income looks potentially heteroskedastic; let's test via Goldfeld-Quandt.

```r
# Arrange the data by income
test_df = arrange(test_df, income)
# Re-estimate the model for the last and first 158 observations
est_model1 = lm(test_score ~ ratio + income, data = tail(test_df, 158))
est_model2 = lm(test_score ~ ratio + income, data = head(test_df, 158))
```

Testing for heteroskedasticity

Example: Goldfeld-Quandt

Income looks potentially heteroskedastic; let's test via Goldfeld-Quandt.

```
# Arrange the data by income
test_df = arrange(test_df, income)
# Re-estimate the model for the last and first 158 observations
est_model1 = lm(test_score ~ ratio + income, data = tail(test_df, 158))
est_model2 = lm(test_score ~ ratio + income, data = head(test_df, 158))
# Grab the residuals from each regression
e_model1 = residuals(est_model1)
e_model2 = residuals(est_model2)
```

Testing for heteroskedasticity

Example: Goldfeld-Quandt

Income looks potentially heteroskedastic; let's test via Goldfeld-Quandt.

```
# Arrange the data by income
test_df = arrange(test_df, income)
# Re-estimate the model for the last and first 158 observations
est_model1 = lm(test_score ~ ratio + income, data = tail(test_df, 158))
est_model2 = lm(test_score ~ ratio + income, data = head(test_df, 158))
# Grab the residuals from each regression
e_model1 = residuals(est_model1)
e_model2 = residuals(est_model2)
# Calculate SSE for each regression
(sse_model1 = sum(e_model1^2))
```

```
#> [1] 19305.01
```

```
(sse_model2 = sum(e_model2^2))
```

```
#> [1] 29537.83
```

Testing for heteroskedasticity

Example: Goldfeld-Quandt

Remember the Goldfeld-Quandt test statistic?

$$F_{n^\star - k,\, n^\star - k} = \frac{\text{SSE}_2}{\text{SSE}_1}$$

Testing for heteroskedasticity

Example: Goldfeld-Quandt

Remember the Goldfeld-Quandt test statistic?

$$F_{n^\star-k,\, n^\star-k} = \frac{\text{SSE}_2}{\text{SSE}_1} \approx \frac{29,537.83}{19,305.01}$$

Testing for heteroskedasticity

Example: Goldfeld-Quandt

Remember the Goldfeld-Quandt test statistic?

$$F_{n^\star-k,\, n^\star-k} = \frac{\text{SSE}_2}{\text{SSE}_1} \approx \frac{29,537.83}{19,305.01} \approx 1.53$$

Testing for heteroskedasticity

Example: Goldfeld-Quandt

Remember the Goldfeld-Quandt test statistic?

$$F_{n^\star-k,\, n^\star-k} = \frac{\text{SSE}_2}{\text{SSE}_1} \approx \frac{29{,}537.83}{19{,}305.01} \approx 1.53 \quad \text{Test via } F_{158-3,\, 158-3}$$

Testing for heteroskedasticity

Example: Goldfeld-Quandt

Remember the Goldfeld-Quandt test statistic?

$$F_{n^\star-k,\, n^\star-k} = \dfrac{\text{SSE}_2}{\text{SSE}_1} \approx \dfrac{29,537.83}{19,305.01} \approx \mathbf{1.53} \quad \text{Test via } F_{158-3,\, 158-3}$$

```
# G-Q test statistic
(f_gq = sse_model2/sse_model1)
```

```
#> [1] 1.530061
```

Testing for heteroskedasticity

Example: Goldfeld-Quandt

Remember the Goldfeld-Quandt test statistic?

$$F_{n^\star-k,\,n^\star-k} = \dfrac{\text{SSE}_2}{\text{SSE}_1} \approx \dfrac{29,537.83}{19,305.01} \approx 1.53 \quad \text{Test via } F_{158-3,\,158-3}$$

```r
# G-Q test statistic
(f_gq = sse_model2/sse_model1)
```

```
#> [1] 1.530061
```

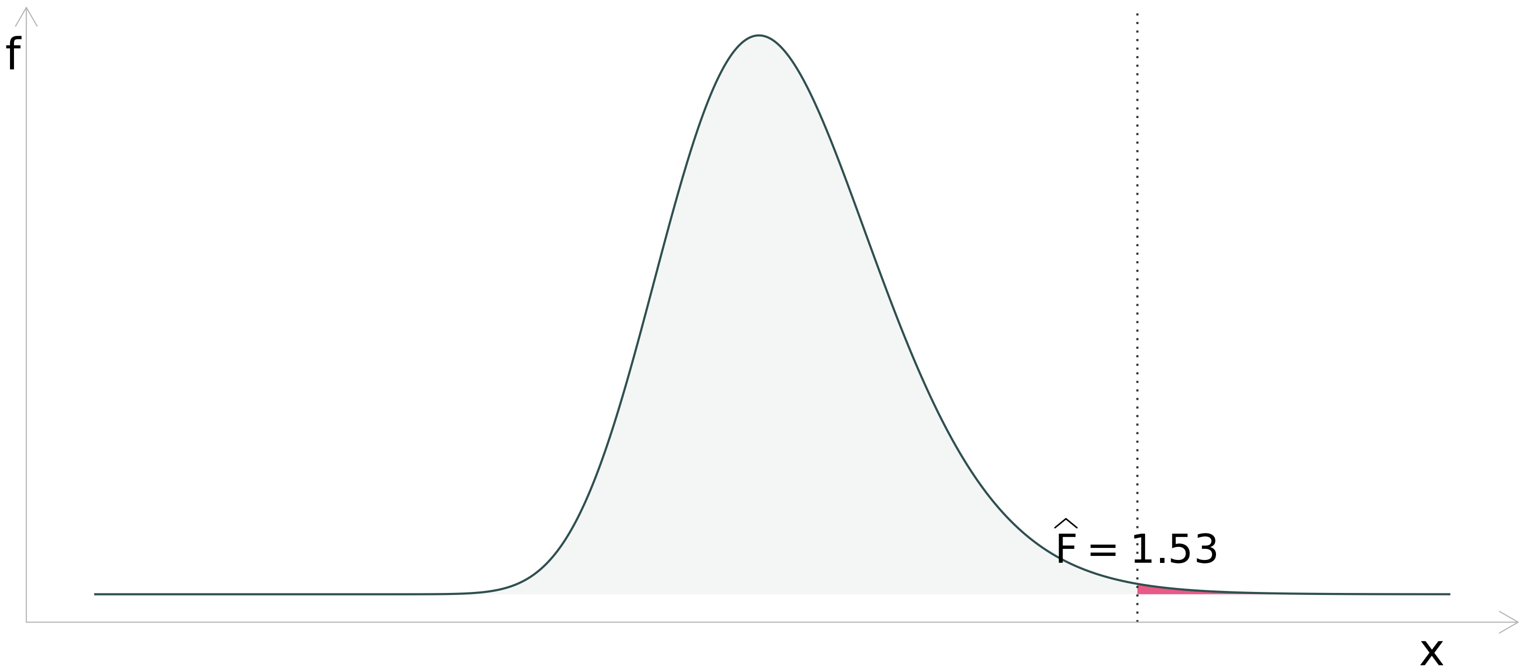
```r
# p-value
pf(q = f_gq, df1 = 158-3, df2 = 158-3, lower.tail = F)
```

```
#> [1] 0.004226666
```

Testing for heteroskedasticity

Example: Goldfeld-Quandt

The Goldfeld-Quandt test statistic and its null distribution



Example: Goldfeld-Quandt

Putting it all together:

H$_0$: $\sigma_1^2 = \sigma_2^2$ *vs.* H$_A$: $\sigma_1^2 \neq \sigma_2^2$

Goldfeld-Quandt test statistic: $F \approx 1.53$

p-value ≈ 0.00423

$\therefore$ Reject H$_0$ (*p*-value is less than 0.05).

Conclusion: There is statistically significant evidence that $\sigma_1^2 \neq \sigma_2^2$. Therefore, we find statistically significant evidence of heteroskedasticity (at the 5-percent level).

Testing for heteroskedasticity

Example: Goldfeld-Quandt

What if we had chosen to focus on student-to-teacher ratio?

Testing for heteroskedasticity

Example: Goldfeld-Quandt

What if we had chosen to focus on student-to-teacher ratio?

```
# Arrange the data by ratio
test_df = arrange(test_df, ratio)
# Re-estimate the model for the last and first 158 observations
est_model3 = lm(test_score ~ ratio + income, data = tail(test_df, 158))
est_model4 = lm(test_score ~ ratio + income, data = head(test_df, 158))
# Grab the residuals from each regression
e_model3 = residuals(est_model3)
e_model4 = residuals(est_model4)
# Calculate SSE for each regression
(sse_model3 = sum(e_model3^2))
```

```
#> [1] 26243.52
```

```
(sse_model4 = sum(e_model4^2))
```

```
#> [1] 29101.52
```

Testing for heteroskedasticity

Example: Goldfeld-Quandt

$$F_{n^\star-k,\,n^\star-k} = \frac{\text{SSE}_4}{\text{SSE}_3} \approx \frac{29,101.52}{26,243.52} \approx 1.11$$

which has a p-value of approximately 0.2603.

Testing for heteroskedasticity

Example: Goldfeld-Quandt

$$F_{n^\star-k,\, n^\star-k} = \dfrac{\text{SSE}_4}{\text{SSE}_3} \approx \dfrac{29,101.52}{26,243.52} \approx 1.11$$

which has a *p*-value of approximately 0.2603.

$\therefore$ We would have failed to reject H_0, concluding that we failed to find statistically significant evidence of heteroskedasticity.

Testing for heteroskedasticity

Example: Goldfeld-Quandt

$$F_{n^\star-k,\, n^\star-k} = \frac{\text{SSE}_4}{\text{SSE}_3} \approx \frac{29,101.52}{26,243.52} \approx 1.11$$

which has a *p*-value of approximately 0.2603.

$\therefore$ We would have failed to reject H_0, concluding that we failed to find statistically significant evidence of heteroskedasticity.

Lesson: Understand the limitations of estimators, tests, *etc.*

Example: White

Let's test the same model and data with the White test.

Recall: We saved our residuals as `e` in our dataset, *i.e.*,

```r
# Estimate the model
est_model = lm(test_score ~ ratio + income, data = test_df)
# Add the residuals to our dataset
test_df$e = residuals(est_model)
```

Example: White

The **White test** adds squared terms and interactions to initial regression specification (the right-hand side)

$$
\begin{aligned}
u_i^2 = &\alpha_0 + \alpha_1 \text{Ratio}_i + \alpha_2 \text{Income}_i \\
&+ \alpha_3 \text{Ratio}_i^2 + \alpha_4 \text{Income}_i^2 + \alpha_5 \text{Ratio}_i \times \text{Income}_i \\
&+ w_i
\end{aligned}
$$

The **White test** tests the null hypothesis
H_0: $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 0$

We just need to write some **R** code to test H_0.

Example: White

Aside: R has funky notation for squared terms and interactions in `lm()`:

- **Squared terms** use `I()`, *e.g.*, `lm(y ~ I(x^2))`
- **Interactions** use `:` between the variables, *e.g.*, `lm(y ~ x1:x2)`

Example: Regress `y` on quadratic of `x1` and `x2`:

```
# Pretend quadratic regression w/ interactions
lm(y ~ x1 + x2 + I(x1^2) + I(x2^2) + x1:x2, data = pretend_df)
```

Example: White

Step 1: Regress e_i^2 on 1st degree, 2nd degree, and interactions

```
# Regress squared residuals on quadratic of explanatory variables
white_model = lm(
  I(e^2) ~ ratio + income + I(ratio^2) + I(income^2) + ratio:income,
  data = test_df
)
# Grab the R-squared
(white_r2 = summary(white_model)$r.squared)
```

Example: White

Step 2: Collect R_e^2 from the regression.

```r
# Regress squared residuals on quadratic of explanatory variables
white_model = lm(
  I(e^2) ~ ratio + income + I(ratio^2) + I(income^2) + ratio:income,
  data = test_df
)
# Grab the R-squared
(white_r2 = summary(white_model)$r.squared)
```

```
#> [1] 0.07332222
```

Example: White

Step 3: Calculate White test statistic $\text{LM} = n \times R_e^2 \approx 420 \times 0.073$

```r
# Regress squared residuals on quadratic of explanatory variables
white_model = lm(
  I(e^2) ~ ratio + income + I(ratio^2) + I(income^2) + ratio:income,
  data = test_df
)
# Grab the R-squared
white_r2 = summary(white_model)$r.squared
# Calculate the White test statistic
(white_stat = 420 * white_r2)
```

```
#> [1] 30.79533
```

Example: White

Step 4: Calculate the associated p-value (where $\mathbf{LM} \overset{d}{\sim} \chi_k^2$); here, $k = 5$

```
# Regress squared residuals on quadratic of explanatory variables
white_model = lm(
  I(e^2) ~ ratio + income + I(ratio^2) + I(income^2) + ratio:income,
  data = test_df
)
# Grab the R-squared
white_r2 = summary(white_model)$r.squared
# Calculate the White test statistic
white_stat = 420 * white_r2
# Calculate the p-value
pchisq(q = white_stat, df = 5, lower.tail = F)
```

```
#> [1] 1.028039e-05
```

Heteroskedasticity

Example: White

Putting everything together...

Heteroskedasticity

Example: White

Putting everything together...

H$_0$: $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 0$

Example: White

Putting everything together...

H_0: $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 0$ *vs.* H_A: $\alpha_i \neq 0$ for some $i \in \{1, 2, \ldots, 5\}$

Heteroskedasticity

Example: White

Putting everything together...

H$_0$: $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 0$ *vs.* H$_A$: $\alpha_i \neq 0$ for some $i \in \{1, 2, \ldots, 5\}$

$$\begin{aligned} u_i^2 = & \alpha_0 + \alpha_1 \text{Ratio}_i + \alpha_2 \text{Income}_i \\ & + \alpha_3 \text{Ratio}_i^2 + \alpha_4 \text{Income}_i^2 \\ & + \alpha_5 \text{Ratio}_i \times \text{Income}_i + w_i \end{aligned}$$

Example: White

Putting everything together...

H$_0$: $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 0$ *vs.* H$_A$: $\alpha_i \neq 0$ for some $i \in \{1, 2, \ldots, 5\}$

$$
\begin{aligned}
u_i^2 =& \alpha_0 + \alpha_1 \text{Ratio}_i + \alpha_2 \text{Income}_i \\
&+ \alpha_3 \text{Ratio}_i^2 + \alpha_4 \text{Income}_i^2 \\
&+ \alpha_5 \text{Ratio}_i \times \text{Income}_i + w_i
\end{aligned}
$$

Our White test statistic: $\text{LM} = n \times R_e^2 \approx 420 \times 0.073 \approx 30.8$

Example: White

Putting everything together...

H_0: $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 0$ *vs.* H_A: $\alpha_i \neq 0$ for some $i \in \{1,\, 2,\, \ldots,\, 5\}$

$$\begin{aligned} u_i^2 = & \alpha_0 + \alpha_1 \text{Ratio}_i + \alpha_2 \text{Income}_i \\ & + \alpha_3 \text{Ratio}_i^2 + \alpha_4 \text{Income}_i^2 \\ & + \alpha_5 \text{Ratio}_i \times \text{Income}_i + w_i \end{aligned}$$

Our White test statistic: $\text{LM} = n \times R_e^2 \approx 420 \times 0.073 \approx 30.8$

Under the χ_5^2 distribution, this $\widehat{\text{LM}}$ has a *p*-value less than 0.001.

Example: White

Putting everything together...

H_0: $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 0$ *vs.* H_A: $\alpha_i \neq 0$ for some $i \in \{1,\, 2,\, \dots,\, 5\}$

$$\begin{aligned} u_i^2 = &\alpha_0 + \alpha_1 \text{Ratio}_i + \alpha_2 \text{Income}_i \\ &+ \alpha_3 \text{Ratio}_i^2 + \alpha_4 \text{Income}_i^2 \\ &+ \alpha_5 \text{Ratio}_i \times \text{Income}_i + w_i \end{aligned}$$

Our White test statistic: $\text{LM} = n \times R_e^2 \approx 420 \times 0.073 \approx 30.8$

Under the χ_5^2 distribution, this $\widehat{\text{LM}}$ has a *p*-value less than 0.001.

$\therefore$ We **reject H_0**

Example: White

Putting everything together...

H_0: $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 0$ *vs.* H_A: $\alpha_i \neq 0$ for some $i \in \{1,\, 2,\, \dots,\, 5\}$

$$\begin{aligned} u_i^2 =& \alpha_0 + \alpha_1 \text{Ratio}_i + \alpha_2 \text{Income}_i \\ &+ \alpha_3 \text{Ratio}_i^2 + \alpha_4 \text{Income}_i^2 \\ &+ \alpha_5 \text{Ratio}_i \times \text{Income}_i + w_i \end{aligned}$$

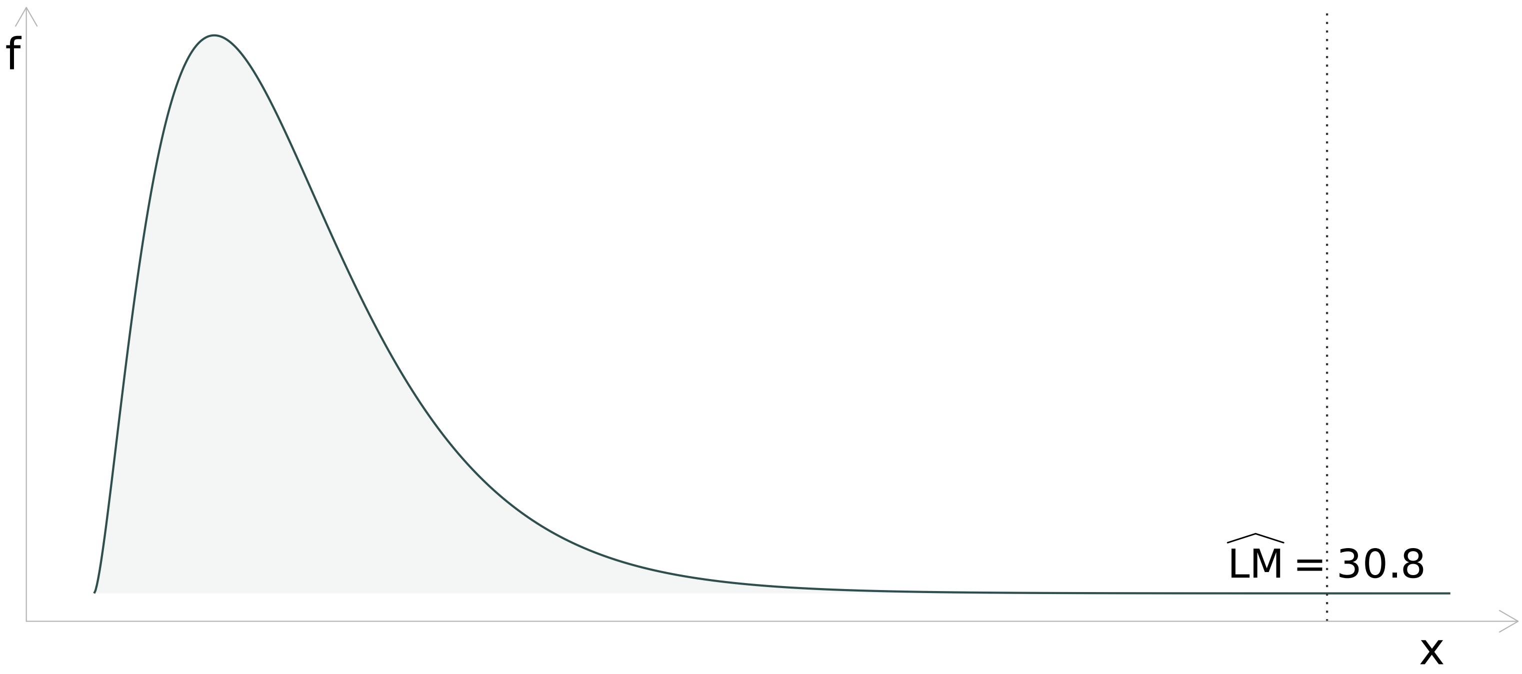
Our White test statistic: $\text{LM} = n \times R_e^2 \approx 420 \times 0.073 \approx 30.8$

Under the χ_5^2 distribution, this $\widehat{\text{LM}}$ has a *p*-value less than 0.001.

∴ We **reject H_0** and conclude there is **statistically significant evidence of heteroskedasticity** (at the 5-percent level).

Example: White

The White test statistic and its null distribution



Heteroskedasticity

Review questions

Heteroskedasticity

Review questions

- **Q:** What is the definition of heteroskedasticity?
- **Q:** Why are we concerned about heteroskedasticity?
- **Q:** Does plotting y against x, tell us anything about heteroskedasticity?
- **Q:** Does plotting e against x, tell us anything about heteroskedasticity?
- **Q:** Since we cannot observe the u_i's, what do we use to *learn about* heteroskedasticity?
- **Q:** Which test do you recommend to test for heteroskedasticity? Why?

Heteroskedasticity

Review questions

- **Q:** What is the definition of heteroskedasticity?

Heteroskedasticity

Review questions

- **Q:** What is the definition of heteroskedasticity?
- **A:**
  Math: $\mathrm{Var}(u_i|X) \neq \mathrm{Var}\big(u_j|X\big)$ for some $i \neq j$.
  Words: There is a systematic relationship between the variance of u_i and our explanatory variables.

Heteroskedasticity

Review questions

- **Q:** What is the definition of heteroskedasticity?

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Review questions

- **Q:** What is the definition of heteroskedasticity?

- **Q:** Why are we concerned about heteroskedasticity?
- **A:** It biases our standard errors—wrecking our statistical tests and confidence intervals. Also: OLS is no longer the most efficient (best) linear unbiased estimator.

Heteroskedasticity

Review questions

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Review questions

- **Q:** What is the definition of heteroskedasticity?

- **Q:** Why are we concerned about heteroskedasticity?

- **Q:** Does plotting y against x, tell us anything about heteroskedasticity?
- **A:** It's not exactly what we want, but since y is a function of x and u, it can still be informative. If y becomes more/less disperse as x changes, we likely have heteroskedasticity.

Review questions

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- **Q:** Does plotting e against x, tell us anything about heteroskedasticity?

Heteroskedasticity

Review questions

- **Q:** What is the definition of heteroskedasticity?
- **Q:** Why are we concerned about heteroskedasticity?
- **Q:** Does plotting y against x, tell us anything about heteroskedasticity?
- **Q:** Does plotting e against x, tell us anything about heteroskedasticity?
- **A:** Yes. The spread of e depicts its variance—and tells us something about the variance of u. Trends in this variance, along x, suggest heteroskedasticity.

Heteroskedasticity

Review questions

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Review questions

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- **Q:** Why are we concerned about heteroskedasticity?
- **Q:** Does plotting y against x, tell us anything about heteroskedasticity?
- **Q:** Does plotting e against x, tell us anything about heteroskedasticity?
- **Q:** Since we cannot observe the u_i's, what do we use to *learn about* heteroskedasticity?
- **A:** We use the e_i's to predict/learn about the u_i's. This trick is key for almost everything we do with heteroskedasticity testing/correction.

Heteroskedasticity

Review questions

- **Q:** What is the definition of heteroskedasticity?
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- **Q:** Which test do you recommend to test for heteroskedasticity? Why?

Heteroskedasticity

Review questions

- **Q:** What is the definition of heteroskedasticity?
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- **Q:** Does plotting e against x, tell us anything about heteroskedasticity?
- **Q:** Since we cannot observe the u_i's, what do we use to *learn about* heteroskedasticity?
- **Q:** Which test do you recommend to test for heteroskedasticity? Why?
- **A:** I like White. Fewer assumptions. Fewer issues.

Next time: Living/working with heteroskedasticity.

Appendix

One more test...

Testing for heteroskedasticity

The Breusch-Pagan test

Breusch and Pagan (1981) attempted to solve this issue of being too specific with the functional form of the heteroskedasticity.

- Allows the data to show if/how the variance of u_i correlates with X.
- If σ_i^2 correlates with X, then we have heteroskedasticity.
- Regresses e_i^2 on $X = [1, x_1, x_2, \ldots, x_k]$ and tests for joint significance.

Testing for heteroskedasticity

The Breusch-Pagan test

How to implement:

1. Regress y on an intercept, x_1, x_2, …, x_k.

2. Record residuals e.

3. Regress e^2 on an intercept, x_1, x_2, …, x_k.

$$e_i^2 = \alpha_0 + \alpha_1 x_{1i} + \alpha_2 x_{2i} + \cdots + \alpha_k x_{ki} + v_i$$

4. Record R^2.

5. Test hypothesis H_0: $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$

The Breusch-Pagan test

The B-P test statistic[†] is

$$\mathrm{LM} = n \times R_e^2$$

where R_e^2 is the R^2 from the regression

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Under the null, LM is asymptotically distributed as χ_k^2.

Testing for heteroskedasticity

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Under the null, $\mathbf{LM}$ is asymptotically distributed as χ_k^2.

This test statistic tests H_0: $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$.

Rejecting the null hypothesis implies evidence of heteroskedasticity.

[†]: This specific form of the test statistic actually comes form Koenker (1981).

Testing for heteroskedasticity

The Breusch-Pagan test

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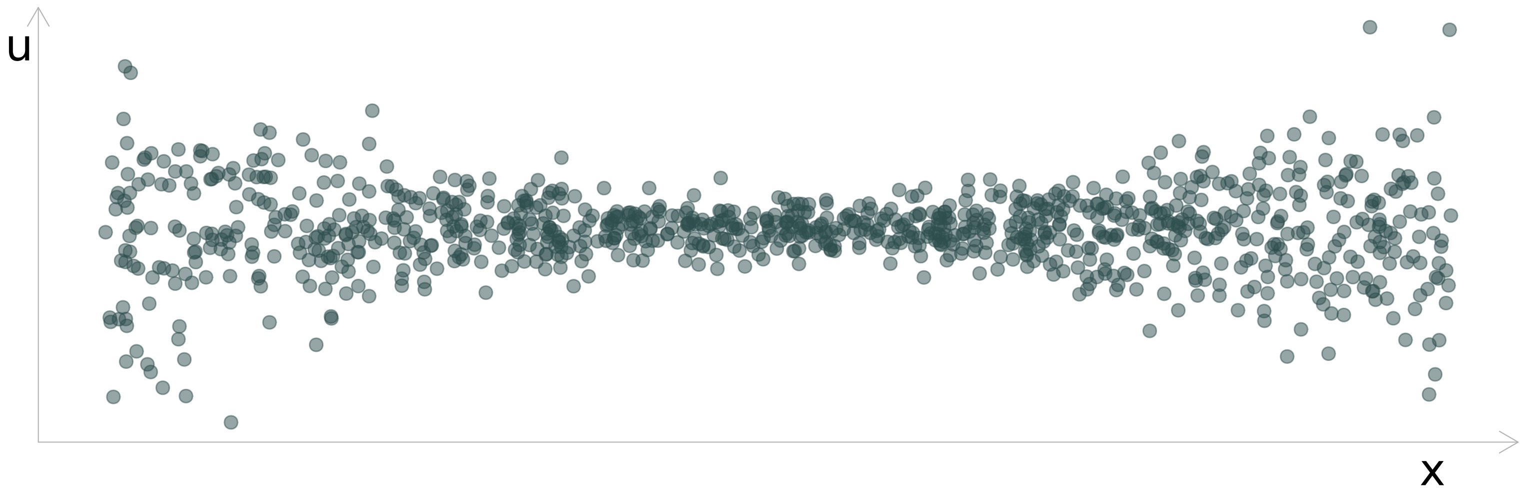
Result: B-P *may* still miss fairly simple forms of heteroskedasticity.

The Breusch-Pagan test

Breusch-Pagan tests are still **sensitive to functional form**.



$$e_i^2 = \hat{\alpha}_0 + \hat{\alpha}_1 x_{1i} \qquad \widehat{\text{LM}} = 1.26 \qquad p\text{-value} \approx 0.261$$

$$e_i^2 = \hat{\alpha}_0 + \hat{\alpha}_1 x_{1i} + \hat{\alpha}_2 x_{1i}^2 \qquad \widehat{\text{LM}} = 185.8 \qquad p\text{-value} < 0.001$$

Testing for heteroskedasticity

Example: Breusch-Pagan

Let's test the same model with the Breusch Pagan.

Recall: We saved our residuals as `e` in our dataset, *i.e.*,

```
test_df$e = residuals(est_model)
```

Testing for heteroskedasticity

Example: Breusch-Pagan

In B-P, we first regress e_i^2 on the explanatory variables,

```
# Regress squared residuals on explanatory variables
bp_model = lm(I(e^2) ~ ratio + income, data = test_df)
```

Testing for heteroskedasticity

Example: Breusch-Pagan

and use the resulting R^2 to calculate a test statistic.

```
# Regress squared residuals on explanatory variables
bp_model = lm(I(e^2) ~ ratio + income, data = test_df)
# Grab the R-squared
(bp_r2 = summary(bp_model)$r.squared)
```

```
#> [1] 3.23205e-05
```

Testing for heteroskedasticity

Example: Breusch-Pagan

The Breusch-Pagan test statistic is

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$$\mathbf{LM} = \boldsymbol{n \times R_e^2 \approx 420 \times 0.0000323}$$

Testing for heteroskedasticity

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The Breusch-Pagan test statistic is

$$\mathbf{LM} = \boldsymbol{n} \times \boldsymbol{R_e^2} \approx \mathbf{420} \times \mathbf{0.0000323} \approx \mathbf{0.0136}$$

which we test against a χ_k^2 distribution (here: $\boldsymbol{k} = \mathbf{2}$).[†]

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The Breusch-Pagan test statistic is

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which we test against a χ_k^2 distribution (here: $k = 2$).[†]

```r
# B-P test statistic
bp_stat = 420 * bp_r2
# Calculate the p-value
pchisq(q = bp_stat, df = 2, lower.tail = F)
```

```
#> [1] 0.9932357
```

[†]: k is the number of explanatory variables (excluding the intercept).

Testing for heteroskedasticity

Example: Breusch-Pagan

H$_0$: $\alpha_1 = \alpha_2 = 0$ *vs.* H$_A$: $\alpha_1 \neq 0$ and/or $\alpha_2 \neq 0$

for the model $u_i^2 = \alpha_0 + \alpha_1 \text{Ratio}_i + \alpha_2 \text{Income}_i + w_i$

Testing for heteroskedasticity

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Testing for heteroskedasticity

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$\therefore$ Fail to reject H_0 (the *p*-value is greater than 0.05)

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Conclusion: We do not find statistically significant evidence of heteroskedasticity at the 5-percent level.

Testing for heteroskedasticity

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for the model $u_i^2 = \alpha_0 + \alpha_1 \text{Ratio}_i + \alpha_2 \text{Income}_i + w_i$

Breusch-Pagan test statistic: $\widehat{\text{LM}} \approx 0.014$

p-value ≈ 0.993

$\therefore$ Fail to reject H_0 (the *p*-value is greater than 0.05)

Conclusion: We do not find statistically significant evidence of heteroskedasticity at the 5-percent level. (We find no evidence of a *linear* relationship between u_i^2 and the explanatory variables.)

Testing for heteroskedasticity

Example: Breusch-Pagan

The Breusch-Pagan test statistic and its null distribution