Hindman by Combinatorial Forcing

Jakub Smolík

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In this note, we present a simple proof of Hindman's theorem by combinatorial forcing. The argument is due to Jan Hubička, who kindly explained it to me. It is almost identical to the one by Baumgartner [1], but we hope that this exposition is slightly clearer. We also mention some earlier results leading up to Hindman's theorem for completeness.

Earlier Related Results

Denote by $\mathbb{N}$ the set of all positive integers and by ω the set of all natural numbers (including zero). By $[A]^k$ we denote the set of all k-element subsets of a set A.

Theorem 1 (Schur, 1916)**.** *If $\mathbb{N}$ is finitely colored, there exist integers x, y, and z of the same color such that $x + y = z$.*

Proof. Given a coloring χ of $\mathbb{N}$, define a coloring χ' of $[\mathbb{N}]^2$ by $\chi'(\{a, b\}) \coloneqq \chi(|a-b|)$. By Ramsey's theorem, there are integers $a < b < c$ such that $[\{a, b, c\}]^2$ is monochromatic, so $\chi(b-a) = \chi(c-b) = \chi(c-a)$. Put $x \coloneqq b-a$ and $y \coloneqq c-b$. $\square$

Remark. Schur derived this result in order to solve Fermat's last theorem modulo a prime [4]. He showed that for every m and for every sufficiently large prime p, there are numbers $x, y, z \in \mathbb{F}_p$ such that $x^m + y^m \equiv z^m \pmod{p}$.

One way to generalize Schur's theorem is to consider more complex constraints such as $x^2 - y = z$ or $x + y + z = w$. This problem was studied by Rado in his PhD thesis [3]. We state one of his main results.

Theorem 2 (Rado, 1933)**.** *Let $S(x_1, \dots, x_n)$ be a linear homogeneous constraint*

$$c_1 x_1 + c_2 x_2 + \cdots + c_n x_n = 0, \quad c_i \in \mathbb{Z}.$$

Then the two following conditions are equivalent.

(a) *For every finite coloring of $\mathbb{N}$, there are integers $a_1, a_2, \dots, a_n$ of the same color such that $S(a_1, a_2, \dots, a_n)$ holds.*

(b) *Some nonempty subset of the coefficients c_i sums up to zero.*

Another way to generalize Schur's theorem is to demand that some numbers x, y, z have the same color, and all their sums $x + y$, $x + z$, $y + z$, and $x + y + z$ also share that color.

For $X \subseteq \mathbb{N}$, define the set of all finite sums of X to be the set

$$FS(X) \coloneqq \Big\{ \textstyle\sum_{x \in A} x \mid \varnothing \neq A \subseteq X \text{ finite} \Big\}.$$

In the late 1960s, Jon Folkman, Richard Rado, and Jon Henry Sanders independently extended Schur's theorem in the following way:

Theorem 3 (Folkman–Rado–Sanders, c. 1968)**.** *If* $\mathbb{N}$ *is finitely colored, there exist arbitrarily large finite sets* $X \subseteq \mathbb{N}$ *such that* $FS(X)$ *is monochromatic.*

Hindman [2] extended this even further in 1974.

Theorem 4 (Hindman, 1974)**.** *If* $\mathbb{N}$ *is finitely colored, there exists an infinite set* $X \subseteq \mathbb{N}$ *such that* $FS(X)$ *is monochromatic.*

Shortly after, Baumgartner gave a shorter combinatorial proof [1], on which our argument is based.

Proof of Hindman's Theorem

We denote by Fin $\subseteq \mathcal{P}(\omega)$ the set of all finite nonempty subsets of natural numbers. A *block scheme* is a family $\mathcal{B} = \langle B_i \,|\, i \in \omega \rangle$ where $B_i \in$ Fin, and if $i < j$, then $\max B_i < \min B_j$. Such a block scheme will sometimes be written as

$$\mathcal{B} = B_0 \oplus B_1 \oplus B_2 \oplus \cdots .$$

Similarly, if we write $X = B_0 \oplus B_1 \oplus \cdots \oplus B_n$, it denotes that $\max B_i < \min B_j$ whenever $i < j \leq n$, and that $X = \{B_i \,|\, i \leq n\}$. If n is an integer, we denote by $\mathcal{B} \restriction n$ the sub-scheme consisting of blocks $B \in \mathcal{B}$ such that $\min B > n$. For a set $D \subseteq$ Fin, define the set of its finite unions as

$$FU(D) \coloneqq \Big\{ \bigcup A \,|\, \varnothing \neq A \subseteq D \text{ finite} \Big\}.$$

Theorem 5. *If* Fin *is finitely colored, there exists a block scheme* $\mathcal{B}$ *such that* $FU(\mathcal{B})$ *is monochromatic.*

Observe that Theorem 5 implies Theorem 4. Indeed, given a coloring χ of $\mathbb{N}$, define an injection $j\colon$ Fin $\to \mathbb{N}$ by $j(A) \coloneqq \sum_{n \in A} 2^n$, and a coloring χ' of Fin by $\chi'(A) \coloneqq \chi(j(A))$. Note that if A and B are disjoint, then $j(A) + j(B) = j(A \cup B)$. Hence, if $FU(\mathcal{B})$ is monochromatic for a block scheme $\mathcal{B} = \langle B_i \,|\, i \in \omega \rangle$, then $FS(X)$ is monochromatic for the set $X = \{ j(B_i) \,|\, i \in \omega \}$.

Definition 1. We define a relation $\sqsubseteq$ on block schemes by letting $\mathcal{B}' \sqsubseteq \mathcal{B}$ if $\mathcal{B}' \subseteq FU(\mathcal{B})$. We say that $S \subseteq$ Fin is *large* for $\mathcal{B}$ if for every $\mathcal{B}' \sqsubseteq \mathcal{B}$ we have $S \cap FU(\mathcal{B}') \neq \varnothing$.

Before we continue, observe the following:

- $\mathcal{B}' \sqsubseteq \mathcal{B} \implies FU(\mathcal{B}') \subseteq FU(\mathcal{B})$,
- $\sqsubseteq$ is a partial order,
- $FU(\mathcal{B})$ is large for $\mathcal{B}$. In particular, Fin is large for $[\omega]^1$,
- if S is large for $\mathcal{B}$ and $\mathcal{B}' \sqsubseteq \mathcal{B}$, then S is large for $\mathcal{B}'$,
- S is large for $\mathcal{B} \iff S \cap FU(\mathcal{B})$ is large for $\mathcal{B}$.

Lemma 1. *If* S *is large for* $\mathcal{B}$ *and* $S = S_1 \cup S_2 \cup \cdots \cup S_n$*, then there are* $i \in \{1, 2, \dots, n\}$ *and* $\mathcal{B}' \sqsubseteq \mathcal{B}$ *such that* S_i *is large for* $\mathcal{B}'$*.*

Proof. Let $S = S_1 \cup S_2$; the general case follows by induction. If S_1 is not large for $\mathcal{B}$, then there is $\mathcal{B}' \sqsubseteq \mathcal{B}$ such that $S_1 \cap FU(\mathcal{B}') = \varnothing$. If S_2 were not large for $\mathcal{B}'$, then there would be $\mathcal{B}'' \sqsubseteq \mathcal{B}'$ such that $S_2 \cap FU(\mathcal{B}'') = \varnothing$. But then $S \cap FU(\mathcal{B}'') = \varnothing$, contradicting the fact that S is large for $\mathcal{B}$. □

Lemma 2. *If S is large for $\mathcal{B} = \langle B_i \mid i \in \omega \rangle$, then there are $x, y \in FU(\mathcal{B})$ such that $y, x \cup y \in S$ and $\max x < \min y$.*

Proof. Put $D \coloneqq S \cap FU(\mathcal{B})$ and observe that if $y \in D$, then there is $z \in D$ such that $\max y < \min z$. Indeed, suppose that there were $y \in D$ such that for all $z \in D$ we had $\min z \leq \max y$. Then the sub-scheme $\mathcal{B}' \coloneqq \mathcal{B} \upharpoonright \max y$ would contradict S being large for $\mathcal{B}$, since $S \cap FU(\mathcal{B}') = \varnothing$.

Using this, we can construct a block scheme

$$\mathcal{Y} = y_0 \oplus y_0' \oplus y_1 \oplus y_1' \oplus y_2 \oplus y_2' \oplus \cdots \sqsubseteq \mathcal{B},$$

where $y_i, y_i' \in S$ for all $i \in \omega$. Consider the block scheme

$$\mathcal{Y}' \coloneqq (y_0 \cup y_0') \oplus (y_1 \cup y_1') \oplus (y_2 \cup y_2') \oplus \cdots \sqsubseteq \mathcal{B}.$$

Since S is large for $\mathcal{B}$, there is $A \in \mathrm{Fin}$ such that $\bigcup_{i \in A}(y_i \cup y_i') \in S$. Suppose that $A = \{i_0 < i_1 < \cdots < i_n < r\}$ and put $x \coloneqq y_r \cup \bigcup_{j \leq n}(y_{i_j} \cup y_{i_j}')$ and $y \coloneqq y_r'$. ☐

Lemma 3. *If S is large for $\mathcal{B}$, then there exists $x \in FU(\mathcal{B})$ such that the set*

$$S(x) \coloneqq \{y \in FU(\mathcal{B}) \mid y, x \cup y \in S \text{ and } \max x < \min y\}$$

is large for some $\mathcal{B}' \sqsubseteq \mathcal{B}$.

Proof. Since Fin is countable, we can enumerate $FU(\mathcal{B})$ as $\langle x_i \mid i \in \omega \rangle$ such that $i < j \implies \max x_i \leq \max x_j$. Assume for contradiction that the claim is false and build a sequence of block schemes $\mathcal{B}_0 \sqsubseteq \mathcal{B}$ and $\mathcal{B}_{i+1} \sqsubseteq \mathcal{B}_i$ that satisfy

(i) $FU(\mathcal{B}_i) \cap S(x_i) = \varnothing$, and

(ii) if $z \in \mathcal{B}_i$ and $\min z \leq \max x_{i+1}$, then $z \in \mathcal{B}_{i+1}$.

Suppose that such a sequence exists. Property (ii) ensures that it has a limit $\mathcal{B}^* \sqsubseteq \mathcal{B}$ consisting of blocks B such that $B \in \mathcal{B}_i$ for all sufficiently large i. But since $FU(\mathcal{B}^*) \cap S(x_i) = \varnothing$ for all $i \in \omega$ due to property (i), the existence of this block scheme contradicts Lemma 2.

We construct the sequence as follows. Since $S(x_0)$ is not large for $\mathcal{B}$, there is $\mathcal{B}_0 \sqsubseteq \mathcal{B}$ such that $FU(\mathcal{B}_0) \cap S(x_0) = \varnothing$. If $\mathcal{B}_i$ has already been defined, to define $\mathcal{B}_{i+1}$, we first let $\mathcal{B}_i^- \coloneqq \mathcal{B}_i \upharpoonright \max x_{i+1}$. Denote the blocks we deleted by $z_0, z_1, \ldots, z_n$. Since $S(x_{i+1})$ is not large for $\mathcal{B}_i^-$, there is $\mathcal{B}_{i+1}' \sqsubseteq \mathcal{B}_i^-$ such that $FU(\mathcal{B}_{i+1}') \cap S(x_{i+1}) = \varnothing$. Define $\mathcal{B}_{i+1}$ as the block scheme obtained from $\mathcal{B}_{i+1}'$ by adding the blocks $z_0, z_1, \ldots, z_n$ back. Note that still $FU(\mathcal{B}_{i+1}) \cap S(x_{i+1}) = \varnothing$ because $\min z_j \leq \max x_{i+1}$ for all $j \leq n$. ☐

Lemma 4. *If S is large for $\mathcal{B}$, then there is $\mathcal{B}^* \sqsubseteq \mathcal{B}$ such that $FU(\mathcal{B}^*) \subseteq S$.*

Proof. First, construct a block scheme $\mathcal{X} = x_0 \oplus x_1 \oplus x_2 \oplus \cdots \sqsubseteq \mathcal{B}$ by repeated application of Lemma 3. We do this by building sequences x_n, $\mathcal{B}_n$, and S_n where

(1) $\mathcal{B}_0 = \mathcal{B}$ and $S_0 = S$,

(2) $\mathcal{B}_{n+1} \sqsubseteq \mathcal{B}_n$ and $S_{n+1} \subseteq S_n \cap FU(\mathcal{B}_n)$,

(3) S_n is large for $\mathcal{B}_n$,

(4) $x_n \in FU(\mathcal{B}_n)$, and if $y \in S_{n+1}$, then $x_n \cup y \in S_n$ and $\max x_n < \min y$.

By combining properties (2) and (4), we get that

(5) if $y \in S_m$ where $m > n$, then $x_n \cup y \in S_n$.

Next, construct a block scheme

$$\mathcal{B}^* = B_0 \oplus B_1 \oplus B_2 \oplus \cdots \sqsubseteq \mathcal{X}.$$

Choose $B_0 \in S \cap FU(\mathcal{X})$ arbitrarily (this is possible since S is large for $\mathcal{B}$), and for $n > 1$ select $B_n \in S_{k_n+1} \cap FU(\mathcal{X})$ where $k_n := \max\{k \mid x_k \subseteq B_{n-1}\}$. This is possible because $\mathcal{X} \upharpoonright \max x_{k_n} \sqsubseteq \mathcal{B}_{k_n+1}$, and S_{k_n+1} is large for $\mathcal{B}_{k_n+1}$. Note that $\max B_{n-1} < \min B_n$ since $\max B_{n-1} = \max x_{k_n}$.

We claim that $\mathcal{B}^*$ has the desired property. Let $A \in FU(\mathcal{B}^*)$ and suppose that

$$A = B_{i_1} \oplus B_{i_2} \oplus \cdots \oplus B_{i_n} \oplus B_r.$$

for some indices $i_1 < i_2 < \cdots < i_n < r$. Since $\mathcal{B}^* \sqsubseteq \mathcal{X}$, we can write

$$B_{i_1} \oplus B_{i_2} \oplus \cdots \oplus B_{i_n} = x_{j_1} \oplus x_{j_2} \oplus \cdots \oplus x_{j_m}$$

for some indices $j_1 < j_2 < \cdots < j_m$. Note that $j_m \leq k_r$. Since $B_r \in S_{k_r+1} \subseteq S_{j_m+1}$, we have $x_{j_m} \cup B_r \in S_{j_m}$ by property (5). Thus $x_{j_{m-1}} \cup x_{j_m} \cup B_r \in S_{j_{m-1}}$ again by property (5) because $j_m > j_{m-1}$. By repeating this, we see that

$$A = x_{j_1} \cup \cdots \cup x_{j_{m-1}} \cup x_{j_m} \cup B_r \in S_{j_1} \subseteq S. \qquad \square$$

We can now finish the proof of Hindman's theorem.

Proof of Theorem 5. Let Fin be r-colored; that is, partitioned into r classes

$$\mathrm{Fin} = S_1 \cup S_2 \cup \cdots \cup S_r.$$

Since Fin is large for $[\omega]^1$, by Lemma 1 there are i and $\mathcal{B} \sqsubseteq [\omega]^1$ such that S_i is large for $\mathcal{B}$. By Lemma 4, there is $\mathcal{B}^* \sqsubseteq \mathcal{B}$ such that $FU(\mathcal{B}^*) \subseteq S_i$. $\square$

Remarks

Hindman's theorem implies all the lemmas we used to prove it. Therefore, if one believes that Hindman's theorem is true, they must also believe in these lemmas, even before proving them.

To demonstrate this, we will show that Theorem 5 implies Lemma 4. Suppose that S is large for $\mathcal{B} = \langle B_i \mid i \in \omega \rangle$. Color $A \in \mathrm{Fin}$ red if $g(A) := \bigcup_{i \in A} B_i \in S$, and blue otherwise. By Theorem 5, there is a homogeneous block scheme

$$\mathcal{X} = \langle x_n \mid n \in \omega \rangle \sqsubseteq [\omega]^1.$$

It defines a block scheme $\mathcal{B}^* = \langle D_n \mid n \in \omega \rangle \sqsubseteq \mathcal{B}$ where $D_n = g(x_n)$. If $FU(\mathcal{X})$ is red, then $FU(\mathcal{B}^*) \subseteq S$, and we are happy. If $FU(\mathcal{X})$ is blue, then $FU(\mathcal{B}^*) \cap S = \varnothing$, contradicting the fact that S is large for $\mathcal{B}$.

Bibliography

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