Lecture 11

Spatial Models

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Roadmap

1. Spatial models: Armington
2. Exact hat algebra
3. Dynamic spatial models: Armington + migration
4. Dynamic hat algebra

Spatial models

So far we have been focusing on **dynamics**

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Now we are going to look at the other dimension: **space**

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To start, we will work with static models

But we will introduce multiple regions across space, and frictions inhibiting mobility of goods and factors of production

The Armington model

Armington and gravity

Classic trade theory (e.g. Ricardo) highlights economic forces driving trade of goods

- Comparative advantage

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Empirical work has long used the **gravity model:**

$$X_{ij} = \alpha \frac{Y_i Y_j}{D_{ij}}$$

Armington and gravity

$$X_{ij} = \alpha \frac{Y_i Y_j}{D_{ij}}$$

X_{ij} trade flow from i to j

Y_i GDP of origin

Y_j GDP of destination

D_{ij} distance and other frictions affecting trade flows

Armington and gravity

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The **Armington** model provides a simple theoretical foundation for gravity with two key ingredients:

1. Spatially differentiated products
2. Love-of-variety preferences

Armington

The set up:

N regions

Armington

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Each region produces a differentiated product

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Representative household in each region can purchase goods from all locations

Trade frictions (e.g. distance) result in different prices offered by different producers

Armington: preferences

The representative household in region j has CES preferences across goods:

$$U_j = \left(\sum_{i \in N} a_{ij}^{\frac{1}{\sigma}} q_{ij}^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}$$

σ is the elasticity of substitution with $\sigma > 1$

a_{ij} is an exogenous taste parameter

q_{ij} is the quantity of goods imported from i to j

Armington: demand

The consumer maximizes utility subject to a budget:

$$\max_{\{q_{ij}\}_{i \in N}} \left(\sum_{i \in N} a_{ij}^{\frac{1}{\sigma}} q_{ij}^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} \quad \text{subject to:} \quad \sum_{i \in N} q_{ij} p_{ij} \leq Y_j$$

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Standard result gives demand for goods from i by j:

$$q_{ij} = a_{ij} p_{ij}^{-\sigma} Y_j P_j^{\sigma-1} \quad \text{where} \quad P_j = \left(\sum_{k\in N} a_{kj} p_{kj}^{1-\sigma}\right)^{\frac{1}{1-\sigma}}$$

P_j is the usual Dixit-Stiglitz price index

Armington: demand

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Multiply by prices to get trade flows in dollar terms:

$$X_{ij} = q_{ij} p_{ij} = a_{ij} p_{ij}^{1-\sigma} Y_j P_j^{\sigma-1}$$

Trade flows **decrease** in bilateral prices, **increase** in the local price index, and **increase** in local size/GDP

Armington: supply

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Market structure:

- Producers compete in perfect competition
- Regions are endowed with L_i workers supplying 1 unit of labor
- Each unit of labor can produce A_i units so total output is $A_i L_i$
- Workers are paid a wage w_i so that $Y_i = w_i L_i$

Armington: supply

Given a labor endowment L_i and productivity A_i the marginal cost of production is $\frac{w_i}{A_i}$

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The price in j to buy a unit of the good from i is then:

$$p_{ij} = \tau_{ij} \frac{w_i}{A_i}$$

Plug back into demand to get:

$$X_{ij} = a_{ij} \tau_{ij}^{1-\sigma} \left(\frac{w_i}{A_i} \right)^{1-\sigma} Y_j P_j^{\sigma-1}$$

Armington: market clearing

$$X_{ij} = a_{ij}\tau_{ij}^{1-\sigma}\left(\frac{w_i}{A_i}\right)^{1-\sigma} Y_j P_j^{\sigma-1}$$

Define **expenditure shares** as the expenditures on i by j relative to js total expenditures:

$$\lambda_{ij} = \frac{X_{ij}}{\sum_{k \in N} X_{kj}}$$

Armington: market clearing

$$\lambda_{ij} = \frac{X_{ij}}{\sum_{k \in N} X_{kj}}$$

$$= \frac{a_{ij}\tau_{ij}^{1-\sigma}\left(\frac{w_i}{A_i}\right)^{1-\sigma} Y_j P_j^{\sigma-1}}{\sum_{k \in N} a_{kj}\tau_{kj}^{1-\sigma}\left(\frac{w_k}{A_k}\right)^{1-\sigma} Y_j P_j^{\sigma-1}}$$

$$= \frac{a_{ij}\left(\frac{\tau_{ij} w_i}{A_i}\right)^{1-\sigma}}{\sum_{k \in N} a_{kj}\left(\frac{\tau_{kj} w_k}{A_k}\right)^{1-\sigma}}$$

Armington: market clearing

$$\lambda_{ij} = \frac{a_{ij}\left(\frac{\tau_{ij} w_i}{A_i}\right)^{1-\sigma}}{\sum_{k \in N} a_{kj}\left(\frac{\tau_{kj} w_k}{A_k}\right)^{1-\sigma}}$$

j spends more on i relative to other places if i has lower wages, higher productivity, or lower trade costs relative to other locations

Armington: market clearing

In perfect competition the expenditures on inputs in j need to match the spending by other locations i on js goods:

$$w_j L_j = \sum_{i \in N} \lambda_{ji} w_i L_i$$

Perfect competition $\rightarrow$ labor costs = revenues

Armington: equilibrium

Our equilibrium is then defined by two sets of equations:

$$\lambda_{ij} = \frac{a_{ij}\left(\frac{\tau_{ij} w_i}{A_i}\right)^{1-\sigma}}{\sum_{k\in N} a_{kj}\left(\frac{\tau_{kj} w_k}{A_k}\right)^{1-\sigma}} \qquad w_j L_j = \sum_{i\in N} \lambda_{ji} w_i L_i$$

where the endogenous variables are the N^2 λ_{ij} terms and the N w_j terms

Armington: equilibrium

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Given the exogenous parameters $a_{ij}, \tau_{ij}, A_i, L_i, \sigma$, how do we solve for the equilibrium?

Armington: equilibrium

$$\lambda_{ij} = \frac{a_{ij}\left(\frac{\tau_{ij}w_i}{A_i}\right)^{1-\sigma}}{\sum_{k\in N} a_{kj}\left(\frac{\tau_{kj}w_k}{A_k}\right)^{1-\sigma}} \qquad w_jL_j = \sum_{i\in N}\lambda_{ji}w_iL_i$$

Given the exogenous parameters $a_{ij}, \tau_{ij}, A_i, L_i, \sigma$, how do we solve for the equilibrium?

We can use function iteration after substituting in for λij in market clearing:

$$w_jL_j = \sum_{i\in N} \frac{a_{ji}\left(\frac{\tau_{ji}w_j}{A_j}\right)^{1-\sigma}}{\sum_{k\in N} a_{ki}\left(\frac{\tau_{ki}w_k}{A_k}\right)^{1-\sigma}} w_iL_i$$

Armington: equilibrium

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We have N market clearing conditions and N unknown w_i terms

Armington: equilibrium

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We have N market clearing conditions and N unknown w_i terms

Write up a function `solve_armington_eq(sigma, tau, A, L, a, tol, damp)` that solves for the equilibrium wages and expenditure shares

Armington: equilibrium

```
function solve_armington_eq(σ, τ, A, L, a, tol = 1e-5, damp = .1)
    w, λ  = ones(size(A)), zeros(size(a))
    wage_error = 1e5
    while wage_error > tol
        denominator, wL = zeros(size(A)), zeros(size(A))
        for k in eachindex(A)
            denominator .+= a[k,:] .* (τ[k,:] * w[k] / A[k]).^(1 .- σ)
        end
        for i in eachindex(A)
            wL .+= a[:,i] .* (τ[:,i] .* w ./ A).^(1 .- σ) .* w[i] .* L[i] ./ denominator[i]
        end
        wnew = wL ./ L
        wage_error = maximum(abs.(wnew .- w)./w)
        w = damp .* wnew .+ (1 - damp) .* w
    end
    for o in eachindex(A), d in eachindex(A)
        λ[o,d] = a[o,d] * (τ[o,d] * w[o] / A[o])^(1 - σ) / sum(a[:,d] .* (τ[:,d] .* w[:] ./ A[:]
    end
    return w, λ
end
```

Symmetric + cross-region trade costs of 5

```
τ = [1. 5; 5. 1.];
a = [1. 1.; 1. 1.];
A = [1., 1.];
L = [1., 1.];
σ = 2.;
w, λ = solve_armington_eq(σ, τ, A, L, a);
display(w)
```

```
## 2-element Vector{Float64}:
##  1.0
##  1.0
```

```
display(λ)
```

```
## 2×2 Matrix{Float64}:
##  0.833333  0.166667
##  0.166667  0.833333
```

Productivity shock to region 1

```
τ = [1. 5; 5. 1.];
a = [1. 1.; 1. 1.];
A = [10., 1.];
L = [1., 1.];
σ = 2.;
w, λ = solve_armington_eq(σ, τ, A, L, a);
display(w)
```

```
## 2-element Vector{Float64}:
##  1.6143181095745787
##  0.3856818904254278 3
```

```
display(λ)
```

```
## 2×2 Matrix{Float64}:
##  0.922754  0.323331
##  0.077246  0.676669
```

Increased labor supply in region 1

```
τ = [1. 5; 5. 1.];
a = [1. 1.; 1. 1.];
A = [1., 1.];
L = [5., 1.];
σ = 2.;
w, λ = solve_armington_eq(σ, τ, A, L, a);
display(w)
```

```
## 2-element Vector{Float64}:
##  0.8780637781792023
##  1.6096811091040044
```

```
display(λ)
```

```
## 2×2 Matrix{Float64}:
##  0.901634   0.26828
##  0.0983663  0.73172
```

Return to autarky

```
τ = [1. 1e9; 1e9 1.];
a = [1. 1.; 1. 1.];
A = [1., 1.];
L = [1., 1.];
σ = 2.;
w, λ = solve_armington_eq(σ, τ, A, L, a);
display(w)
```

```
## 2-element Vector{Float64}:
##  1.0
##  1.0
```

```
display(λ)
```

```
## 2×2 Matrix{Float64}:
##  1.0      1.0e-9
##  1.0e-9   1.0
```

Free trade

```
τ = [1. 1.0001; 1.0001 1.];
a = [1. 1.; 1. 1.];
A = [1., 1.];
L = [1., 1.];
σ = 2.;
w, λ = solve_armington_eq(σ, τ, A, L, a);
display(w)
```

```
## 2-element Vector{Float64}:
##  1.0
##  1.0
```

```
display(λ)
```

```
## 2×2 Matrix{Float64}:
##  0.500025  0.499975
##  0.499975  0.500025
```

New trading partner

```
τ = [1. 2. 3.; 3. 1. 2.; 1e9 5. 1.];
a = [1. 1. 1.; 1. 1. 1.; 1. 1. 1.];
A = [1., 1., 1.];
L = [1., 1., 1.];
σ = 2.;
w, λ = solve_armington_eq(σ, τ, A, L, a);
display(w)
```

```
## 3-element Vector{Float64}:
##  1.2539609132194725
##  1.095427131707213
##  0.6506119550733213
```

```
display(λ)
```

```
## 3×3 Matrix{Float64}:
##  0.723812    0.246282  0.117659
##  0.276188    0.563849  0.20203
##  1.39504e-9  0.189869  0.680311
```

Solving Armington

We solved for the Armington equilibrium

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This is a bit unsatisfying: we'd like to not have to take a stand on numerous region-specific and bilateral parameters

Exact hat algebra

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There are two key pieces to it:

1. Real world data (wages, trade flows) are essentially sufficient statistics for unobservable parameters (productivity, trade costs)
2. Spatial models are built in a way where we can express a counterfactual equilibrium in terms of changes relative to the factual

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2. Spatial models are built in a way where we can express a counterfactual equilibrium in terms of changes relative to the factual

Going forward, primes will indicate counterfactual quantities (w_i'), hats will indicate relative quantities $(\hat{w}_i = \frac{w_i'}{w_i})$

Exact hat algebra

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We want to understand the equilibrium effects of some arbitrary changes in the distribution of productivity: $\{\hat{A}_1, \hat{A}_2, \ldots\}$

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We observe the data for this equilibrium (wages, trade flows, etc)

We want to understand the equilibrium effects of some arbitrary changes in the distribution of productivity: $\{\hat{A}_1, \hat{A}_2, \ldots\}$

Assume no other exogenous variables are changing:
$\hat{\tau}_{ij} = 1, \hat{a}_{ij} = 1, \hat{\sigma} = 1, \hat{L} = 1$

Exact hat algebra

Recall our equilibrium conditions were:

$$\lambda_{ij} = \frac{a_{ij}\left(\frac{\tau_{ij}w_i}{A_i}\right)^{1-\sigma}}{\sum_{k\in N} a_{kj}\left(\frac{\tau_{kj}w_k}{A_k}\right)^{1-\sigma}} \qquad w_j L_j = \sum_{i\in N} \lambda_{ji} w_i L_i$$

Let's start by manipulating the market clearing condition which holds in the factual and counterfactual equilibria:

Exact hat algebra

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Let's start by manipulating the market clearing condition which holds in the factual and counterfactual equilibria:

$$w_j L_j = \sum_{i\in N} \lambda_{ji} w_i L_i \qquad w'_j L'_j = \sum_{i\in N} \lambda'_{ji} w'_i L'_i$$

Exact hat algebra

First by definition: $\lambda_{ji} w_i L_i = X_{ji}$ so that

$$w'_j L'_j = \sum_{i \in N} \lambda'_{ji} w'_i L'_i = \sum_{i \in N} X'_{ji}$$

Exact hat algebra

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$$w'_j L'_j = \sum_{i \in N} \lambda'_{ji} w'_i L'_i = \sum_{i \in N} X'_{ji}$$

Next, put the left hand side into hat form by dividing both sides by $w_j L_j$:

$$\underbrace{\hat{w}_j \hat{L}_j}_{=1} = \sum_{i \in N} \frac{X'_{ji}}{w_j L_j}$$

Exact hat algebra

$$\hat{w}_j = \sum_{i \in N} \frac{X'_{ji}}{w_j L_j}$$

Multiply and divide the right and side by X_{ji} to put it into hat form:

$$\hat{w}_j = \sum_{i \in N} \frac{X_{ji}}{w_j L_j} \hat{X}_{ji}$$

Exact hat algebra

Finally, we know that $X_{ji} = \lambda_{ji} w_i L_i$ so that $\hat{X}_{ji} = \hat{\lambda}_{ji} \hat{w}_i \hat{L}_i$ and

$$\hat{w}_j = \sum_{i \in N} \frac{X_{ji}}{w_j L_j} \hat{\lambda}_{ji} \hat{w}_i$$

The change in wages depends on the change in endogenous wages and expenditure shares, and the observed factual bilateral expenditures, wages, and labor

Exact hat algebra

Now let's go to the gravity equation:

$$\lambda'_{ij} = \frac{a_{ij}\left(\frac{\tau_{ij} w'_i}{A'_i}\right)^{1-\sigma}}{\sum_{k \in N} a_{kj}\left(\frac{\tau_{kj} w'_k}{A'_k}\right)^{1-\sigma}}$$

Exact hat algebra

Now let's go to the gravity equation:

$$\lambda'_{ij} = \frac{a_{ij}\left(\frac{\tau_{ij} w'_i}{A'_i}\right)^{1-\sigma}}{\sum_{k \in N} a_{kj}\left(\frac{\tau_{kj} w'_k}{A'_k}\right)^{1-\sigma}}$$

Put this into hat form:

$$\lambda'_{ij}/\lambda_{ij} = \left[\frac{a_{ij}\left(\frac{\tau_{ij} w'_i}{A'_i}\right)^{1-\sigma}}{\sum_{k \in N} a_{kj}\left(\frac{\tau_{kj} w'_k}{A'_k}\right)^{1-\sigma}}\right] \Bigg/ \left[\frac{a_{ij}\left(\frac{\tau_{ij} w_i}{A_i}\right)^{1-\sigma}}{\sum_{l \in N} a_{lj}\left(\frac{\tau_{lj} w_l}{A_l}\right)^{1-\sigma}}\right]$$

Exact hat algebra

The numerator goes into hats easily, the denominator is trickier:

$$\hat{\lambda}_{ij} = \frac{\left(\frac{\hat{w}_i}{\hat{A}_i}\right)^{1-\sigma}}{\left[\frac{\sum_{k \in N} a_{kj} \left(\frac{\tau_{kj} w'_k}{A'_k}\right)^{1-\sigma}}{\sum_{l \in N} a_{lj} \left(\frac{\tau_{lj} w_l}{A_l}\right)^{1-\sigma}}\right]}$$

Next, bring the bottom sum inside the top sum since it is a function of j and does not depend on k

Exact hat algebra

$$\hat{\lambda}_{ij} = \frac{\left(\frac{\hat{w}_i}{\hat{A}_i}\right)^{1-\sigma}}{\sum_{k \in N} \left[\frac{a_{kj}\left(\frac{\tau_{kj} w'_k}{A'_k}\right)^{1-\sigma}}{\sum_{l \in N} a_{lj}\left(\frac{\tau_{lj} w_l}{A_l}\right)^{1-\sigma}} \right]}$$

Exact hat algebra

$$\hat{\lambda}_{ij} = \frac{\left(\frac{\hat{w}_i}{\hat{A}_i}\right)^{1-\sigma}}{\sum_{k \in N}\left[\frac{a_{kj}\left(\frac{\tau_{kj} w'_k}{A'_k}\right)^{1-\sigma}}{\sum_{l \in N} a_{lj}\left(\frac{\tau_{lj} w_l}{A_l}\right)^{1-\sigma}}\right]}$$

Inside the square brackets, multiply and divide by $a_{kj}\left(\frac{\tau_{kj} w_k}{A_k}\right)^{1-\sigma}$

Exact hat algebra

$$\hat{\lambda}_{ij} = \frac{\left(\frac{\hat{w}_i}{\hat{A}_i}\right)^{1-\sigma}}{\sum_{k\in N}\left[\underbrace{\frac{a_{kj}\left(\frac{\tau_{kj}w'_k}{A'_k}\right)^{1-\sigma}}{a_{kj}\left(\frac{\tau_{kj}w_k}{A_k}\right)^{1-\sigma}}}_{\left(\frac{\hat{w}_k}{\hat{A}_k}\right)^{1-\sigma}}\underbrace{\frac{a_{kj}\left(\frac{\tau_{kj}w_k}{A_k}\right)^{1-\sigma}}{\sum_{l\in N}a_{lj}\left(\frac{\tau_{lj}w_l}{A_l}\right)^{1-\sigma}}}_{\lambda_{kj}}\right]}$$

Exact hat algebra

This finally gives us that:

$$\hat{\lambda}_{ij} = \frac{\left(\frac{\hat{w}_i}{\hat{A}_i}\right)^{1-\sigma}}{\sum_{k \in N} \lambda_{kj} \left(\frac{\hat{w}_k}{\hat{A}_k}\right)^{1-\sigma}}$$

The change in expenditure shares depends on the change in exogenous productivity, endogenous wages, and the observed factual expenditure shares

Exact hat algebra

We now have our two equilibrium conditions in changes:

$$\hat{\lambda}_{ij} = \frac{\left(\frac{\hat{w}_i}{\hat{A}_i}\right)^{1-\sigma}}{\sum_{k \in N} \lambda_{kj} \left(\frac{\hat{w}_k}{\hat{A}_k}\right)^{1-\sigma}} \qquad \hat{w}_j = \sum_{i \in N} \frac{X_{ji}}{w_j L_j} \hat{\lambda}_{ji} \hat{w}_i$$

and can combine them into a single equilibrium condition in changes:

$$\hat{w}_j = \sum_{i \in N} \frac{\frac{X_{ji}}{w_j L_j} \left(\frac{\hat{w}_j}{\hat{A}_j}\right)^{1-\sigma} \hat{w}_i}{\sum_{k \in N} \lambda_{ki} \left(\frac{\hat{w}_k}{\hat{A}_k}\right)^{1-\sigma}}$$

Exact hat algebra

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Notice that it does not depend on any structural parameters except for σ

Exact hat algebra

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$\lambda_{ij}, X_{ij}, w_i, L_i$ are all observable data:

Exact hat algebra

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$\lambda_{ij}, X_{ij}, w_i, L_i$ are all observable data:

$\hat{A}_i$ is a chosen counterfactual

Exact hat algebra

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$\hat{w}_i$ are unknown but can be solved through function iteration

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$$\hat{w}_j = \sum_{i \in N} \frac{\frac{X_{ji}}{w_j L_j} \left(\frac{\hat{w}_j}{\hat{A}_j} \right)^{1-\sigma} \hat{w}_i}{\sum_{k \in N} \lambda_{ki} \left(\frac{\hat{w}_k}{\hat{A}_k} \right)^{1-\sigma}}$$

$\lambda_{ij}, X_{ij}, w_i, L_i$ are all observable data:

$\hat{A}_i$ is a chosen counterfactual

$\hat{w}_i$ are unknown but can be solved through function iteration

We also will want to solve for $\hat{P}_i$ to understand how the price index is changing

Exact hat algebra

Recall:

$$P_j = \left(\sum_{k \in N} a_{kj} p_{kj}^{1-\sigma} \right)^{\frac{1}{1-\sigma}} = \left(\sum_{k \in N} a_{kj} \left(\tau_{kj} \frac{w_k}{A_k} \right)^{1-\sigma} \right)^{\frac{1}{1-\sigma}}$$

$$\hat{P}_j^{1-\sigma} = \frac{\sum_{k \in N} a_{kj} \left(\tau_{kj} \frac{w'_k}{A'_k} \right)^{1-\sigma}}{\sum_{l \in N} a_{lj} \left(\tau_{lj} \frac{w_l}{A_l} \right)^{1-\sigma}}$$

Like before, bring the denominator inside the numerator sum

Exact hat algebra

$$\hat{P}_j^{1-\sigma} = \sum_{k\in N} \frac{a_{kj}\left(\tau_{kj}\frac{w'_k}{A'_k}\right)^{1-\sigma}}{\sum_{l\in N} a_{lj}\left(\tau_{lj}\frac{w_l}{A_l}\right)^{1-\sigma}}$$

multiply and divide by $a_{kj}\left(\tau_{kj}\frac{w_k}{A_k}\right)^{1-\sigma}$ to get:

$$\hat{P}_j^{1-\sigma} = \sum_{k\in N} \frac{a_{kj}\left(\tau_{kj}\frac{w'_k}{A'_k}\right)^{1-\sigma}}{a_{kj}\left(\tau_{kj}\frac{w_k}{A_k}\right)^{1-\sigma}} \frac{a_{kj}\left(\tau_{kj}\frac{w_k}{A_k}\right)^{1-\sigma}}{\sum_{l\in N} a_{lj}\left(\tau_{lj}\frac{w_l}{A_l}\right)^{1-\sigma}}$$

Exact hat algebra

$$\hat{P}_j^{1-\sigma} = \sum_{k \in N} \left(\frac{\hat{w}_k}{\hat{A}_k} \right)^{1-\sigma} \underbrace{\frac{a_{kj} \left(\tau_{kj} \frac{w_k}{A_k} \right)^{1-\sigma}}{\sum_{l \in N} a_{lj} \left(\tau_{lj} \frac{w_l}{A_l} \right)^{1-\sigma}}}_{\lambda_{kj}}$$

$$\hat{P}_j = \left(\sum_{k \in N} \left(\frac{\hat{w}_k}{\hat{A}_k} \right)^{1-\sigma} \lambda_{kj} \right)^{\frac{1}{1-\sigma}}$$

Exact hat algebra

$$\hat{w}_j = \sum_{i \in N} \frac{\frac{X_{ji}}{w_j L_j} \left(\frac{\hat{w}_j}{\hat{A}_j} \right)^{1-\sigma} \hat{w}_i}{\sum_{k \in N} \lambda_{ki} \left(\frac{\hat{w}_k}{\hat{A}_k} \right)^{1-\sigma}}$$

We have N market clearing conditions and N unknown $\hat{w}_i$ terms

Exact hat algebra

$$\hat{w}_j = \sum_{i \in N} \frac{\frac{X_{ji}}{w_j L_j} \left(\frac{\hat{w}_j}{\hat{A}_j} \right)^{1-\sigma} \hat{w}_i}{\sum_{k \in N} \lambda_{ki} \left(\frac{\hat{w}_k}{\hat{A}_k} \right)^{1-\sigma}}$$

We have N market clearing conditions and N unknown $\hat{w}_i$ terms

Write up a function `solve_armington_exact_hat(X, lambda, w, L, Ahat, sigma, tol, damp)` that solves for the new equilibrium in changes

Exact hat algebra

$$\hat{w}_j = \sum_{i \in N} \frac{\frac{X_{ji}}{w_j L_j} \left(\frac{\hat{w}_j}{\hat{A}_j}\right)^{1-\sigma} \hat{w}_i}{\sum_{k \in N} \lambda_{ki} \left(\frac{\hat{w}_k}{\hat{A}_k}\right)^{1-\sigma}}$$

We have N market clearing conditions and N unknown $\hat{w}_i$ terms

Write up a function `solve_armington_exact_hat(X, lambda, w, L, Ahat, sigma, tol, damp)` that solves for the new equilibrium in changes

Key thing to keep in mind: we haven't defined a numeraire yet, use the consumption price index

Exact hat algebra

Here are the data to use:

```
w = [1., 1.];
L = [1., 1.];
λ = [.8 .2; .2 .8];
X = (w .* L)' .* λ;
Ahat = [10., 1.];
σ = 2.;
```

Columns of λ should sum to 1, X is generated to be consistent with w, L, λ

Exact hat algebra

```
function solve_armington_exact_hat(X, λ, w, L, Ahat, σ, tol = 1e-5, damp = .1)
    what = ones(size(Ahat))
    wage_error = 1e5
    while wage_error > tol
        denominator, what_new = zeros(size(Ahat)), zeros(size(Ahat))
        for k in eachindex(Ahat)
            denominator .+= λ[k,:] .* (what[k] / Ahat[k]).^(1 .- σ)
        end
        for i in eachindex(Ahat)
            what_new .+= (X[:,i] ./ (w .* L) ) .* (what ./ Ahat).^(1 .- σ) .* what[i] ./ denomi
        end
        wage_error = maximum(abs.(what_new .- what)./what)
        what = damp .* what_new .+ (1 - damp) .* what
    end
    λhat = (what ./ Ahat).^(1 .- σ) ./ sum(λ .* (what ./ Ahat).^(1 .- σ), dims = 1)
    Phat = vec((sum(λ .* (what ./ Ahat).^(1 .- σ), dims = 1)).^(1 ./ (1 .- σ)))
    return what, λhat, Phat
end
```

```
## solve_armington_exact_hat (generic function with 3 methods)
```

Exact hat algebra

```
Ahat = [10., 1.];
what, λhat, Phat = solve_armington_exact_hat(X, λ, w, L, Ahat, σ);
what ./ Phat
```

```
## 2-element Vector{Float64}:
##  8.817954095849839
##  1.289024997893565
```

When region 1 becomes more productive: their real wages increase > 800%, region 2's real wages increase 30%

Exact hat algebra

```
λhat
```

```
## 2×2 Matrix{Float64}:
##  1.13405  1.89688
##  0.4638   0.77578
```

Both region's expenditures tilt toward region 1

Exact hat algebra

There was nothing special about productivity here

Exact hat algebra

There was nothing special about productivity here

We could have looked at changes in trade costs, preference parameters, labor endowments, or any combination of them

Armington with dynamic migration

Dynamics in spatial models

Introducing meaningful dynamics in spatial models is difficult

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The intuition is that when we introduce additional regions, the state space grows dramatically

This is then exacerbated by introducing meaningful notions of time

How can we begin to introduce some dynamics into spatial models?

One way is to essentially layer a separate, tractable dynamic model onto our static Armington model

Dynamics in spatial models

How we will do this is by introducing dynamic migration decisions of households

Dynamics in spatial models

How we will do this is by introducing dynamic migration decisions of households

First we will introduce static migration to get a sense of how it works

Armington + migration

The set up:

N regions with a measure $L = 1$ total households across all regions

Armington + migration

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Each region produces a differentiated product

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Representative household in each region can purchase goods from all locations

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Representative household in each region can purchase goods from all locations

Trade frictions result in different prices offered by different producers

Armington + migration

The set up:

N regions with a measure $L = 1$ total households across all regions

Each region produces a differentiated product

Representative household in each region can purchase goods from all locations

Trade frictions result in different prices offered by different producers

Households frictionlessly choose where to live to maximize their utility

Armington + migration: indirect utility

The household makes two choices:

1. Which region j to live in subject
2. How to allocate their budget across the menu of possible N goods

Armington + migration: indirect utility

The household makes two choices:

1. Which region j to live in subject
2. How to allocate their budget across the menu of possible N goods

We make two additional tweaks to the model:

1. Adding a **type 1 extreme value**, destination-specific idiosyncratic shock ε_j observed by the households
2. Adding log utility over the CES aggregator

Armington + migration: indirect utility

The consumer maximizes utility subject to their wage w_j:

$$\max_{\{q_{ij}\}_{i\in N}} \log\left[\left(\sum_{i\in N} a_{ij}^{\frac{1}{\sigma}} q_{ij}^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}\right] + \varepsilon_j \quad \text{subject to:} \quad \sum_{i\in N} q_{ij} p_{ij} \leq w_j$$

Armington + migration: indirect utility

We get the standard result for their real wage C_j under CES preferences:

$$C_j = w_j / P_j \quad \text{where} \quad P_j = \left(\sum_{k \in N} a_{kj} p_{kj}^{1-\sigma} \right)^{\frac{1}{1-\sigma}}$$

P_j is the usual Dixit-Stiglitz price index

Armington + migration: indirect utility

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P_j is the usual Dixit-Stiglitz price index

We now have the households' real wage conditional on choosing j, we can now solve for the households' optimal choice of j:

$$\max_{j \in N} \log C_j + \varepsilon_j$$

Armington + migration: labor supply

$$\max_{j \in N} \log C_j + \varepsilon_j$$

The household just chooses the location with the highest combination of real wages C_j

Armington + migration: labor supply

$$\max_{j \in N} \log C_j + \varepsilon_j$$

The household just chooses the location with the highest combination of real wages C_j

How many households choose each location? The Frechet assumption on ε_j buys us a closed form solution (see any treatment on discrete choice models)

Armington + migration: labor supply

If $\varepsilon_j \sim T1EV$, with mean 0 variance 1, the share of the $L = 1$ households choosing to live in region j is:

$$L_j = \frac{\exp \log C_j}{\sum_{k \in N} \exp \log C_k} = \frac{\frac{w_j}{P_j}}{\sum_{k \in N} \frac{w_k}{P_k}}$$

where $\sum_{j \in N} L_j = 1$

Armington + migration: labor supply

If $\varepsilon_j \sim T1EV$, with mean 0 variance 1, the share of the $L = 1$ households choosing to live in region j is:

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This is essentially our extensive margin of labor supply

Armington + migration: labor supply

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where $\sum_{j \in N} L_j = 1$

This is essentially our extensive margin of labor supply

Higher wages or lower prices in j relative to other locations attracts more workers

Armington + migration: equilibrium

We now have two equilibrium conditions, labor supply and joint market clearing:

$$L_j = \frac{\exp \frac{w_j}{P_j}}{\sum_{k \in N} \exp \frac{w_k}{P_k}} \qquad w_j L_j = \sum_{i \in N} \frac{a_{ij} \left(\frac{\tau_{ij} w_i}{A_i} \right)^{1-\sigma}}{\sum_{k \in N} a_{kj} \left(\frac{\tau_{kj} w_k}{A_k} \right)^{1-\sigma}} w_i L_i$$

where:

- $P_j = \left(\sum_{k \in N} a_{kj} \left(\tau_{kj} \frac{w_k}{A_k} \right)^{1-\sigma} \right)^{\frac{1}{1-\sigma}}$
- $\sum_{j \in N} L_j = 1$

Armington + migration: exact hat algebra

Now let's solve the model using exact hat algebra

Armington + migration: exact hat algebra

Now let's solve the model using exact hat algebra

Since labor is endogenous, we now need to account for it in the market clearing condition:

$$\hat{w}_j \hat{L}_j = \sum_{i \in N} \frac{\frac{X_{ji}}{w_j L_j} \left(\frac{\hat{w}_j}{\hat{A}_j} \right)^{1-\sigma} \hat{w}_i \hat{L}_i}{\sum_{k \in N} \lambda_{ki} \left(\frac{\hat{w}_k}{\hat{A}_k} \right)^{1-\sigma}}$$

You can prove to yourself that this is the correct expression

Armington + migration: exact hat algebra

Next we need to put labor supply in hat terms:

$$L_j = \frac{\frac{w_j}{P_j}}{\sum_{k \in N} \frac{w_k}{P_k}}$$

$$\hat{L}_j = \frac{\frac{\hat{w}_j}{\hat{P}_j}}{\frac{\sum_{k \in N} \frac{w'_k}{P'_k}}{\sum_{l \in N} \frac{w_l}{P_l}}}$$

Next, use a similar multiply and divide by $\frac{w_j}{P_j}$ trick as for λ

Armington + migration: exact hat algebra

$$\hat{L}_j = \frac{\frac{\hat{w}_j}{\hat{P}_j}}{\left[\frac{\sum_{k\in N} \frac{w'_k}{P'_k} \frac{\frac{w_k}{P_k}}{\frac{w_k}{P_k}}}{\sum_{l\in N} \frac{w_l}{P_l}}\right]} = \frac{\frac{\hat{w}_j}{\hat{P}_j}}{\left[\frac{\sum_{k\in N} \frac{\hat{w}_k}{\hat{P}_k} \frac{w_k}{P_k}}{\sum_{l\in N} \frac{w_l}{P_l}}\right]} = \frac{\frac{\hat{w}_j}{\hat{P}_j}}{\sum_{k\in N} \frac{\hat{w}_k}{\hat{P}_k} \underbrace{\left[\frac{\frac{w_k}{P_k}}{\sum_{l\in N} \frac{w_l}{P_l}}\right]}_{L_k}}$$

$$\hat{L}_j = \frac{\frac{\hat{w}_j}{\hat{P}_j}}{\sum_{k\in N} L_k \frac{\hat{w}_k}{\hat{P}_k}}$$

The change in labor depends on the change in real wages but also the initial labor allocation

Armington + migration: exact hat algebra

$$\hat{w}_j \hat{L}_j = \sum_{i \in N} \frac{\frac{X_{ji}}{w_j L_j} \left(\frac{\hat{w}_j}{\hat{A}_j} \right)^{1-\sigma} \hat{w}_i \hat{L}_i}{\sum_{k \in N} \lambda_{ki} \left(\frac{\hat{w}_k}{\hat{A}_k} \right)^{1-\sigma}} \qquad \hat{L}_j = \frac{\frac{\hat{w}_j}{\hat{P}_j}}{\sum_{k \in N} L_k \frac{\hat{w}_k}{\hat{P}_k}}$$

We now have our two equilibrium conditions in changes that we can iterate on to recover $\hat{w}_j, \hat{L}_j$

Write up a function `solve_armington_mig_exact_hat(X, lambda, w, L, Ahat, sigma, tol, damp)` that solves for the new equilibrium in changes

Armington + migration: exact hat algebra

Here are the data to use:

```
w = [6., 3.];
L = [.3, .7];
λ = [.8 .2; .2 .8];
X = (w .* L)' .* λ
```

```
## 2×2 Matrix{Float64}:
##  1.44  0.42
##  0.36  1.68
```

```
Ahat = [.5, 1.];
σ = 2.;
```

Columns of λ should sum to 1, L should sum to 1, X is generated to be consistent with w, L, λ

Exact hat algebra

```
function solve_armington_mig_exact_hat(X, λ, w, L, Ahat, σ, tol = 1e-5, damp = .1)
    what, Lhat, labor_error, wage_error = ones(size(Ahat)), ones(size(Ahat)), 1e5, 1e5
    while max(labor_error, wage_error) > tol
        denominator, what_new, Lhat_new = zeros(size(Ahat)), zeros(size(Ahat)), zeros(size(Ahat
        for k in eachindex(Ahat)
            denominator .+= λ[k,:] .* (what[k] / Ahat[k]).^(1 .- σ)
        end
        for i in eachindex(Ahat)
            what_new .+= (X[:,i] ./ (w .* L) ) .* (what ./ Ahat).^(1 .- σ) .* what[i] .* Lhat[i
        end
        Phat = vec((sum(λ .* (what_new ./ Ahat).^(1 .- σ), dims = 1)).^(1 ./ (1 .- σ)))
        Lhat_new = what ./ Phat ./ sum(L .* what ./ Phat)
        wage_error, labor_error = maximum(abs.(what_new .- what)./what), maximum(abs.(Lhat_new
        what = damp .* what_new .+ (1 - damp) .* what
        Lhat = damp .* Lhat_new .+ (1 - damp) .* Lhat
    end
    λhat = (what ./ Ahat).^(1 .- σ) ./ sum(λ .* (what ./ Ahat).^(1 .- σ), dims = 1)
    Phat = vec((sum(λ .* (what ./ Ahat).^(1 .- σ), dims = 1)).^(1 ./ (1 .- σ)))
    return what, λhat, Phat, Lhat
end
```

Exact hat algebra

```
Ahat = [.5, 1.];
what, λhat, Phat, Lhat = solve_armington_mig_exact_hat(X, λ, w, L, Ahat, σ)
```

```
## ([0.9051035127985965, 1.0832507504862192], [0.8816650787044907 0.6506730472117545; 1.473339685182036
```

```
what ./ Phat
```

```
## 2-element Vector{Float64}:
##  0.5671087718872735
##  0.9196825263816277
```

When region 1 becomes less productive by 50%: their real wages fall by about the same amount, region 2s real wages fall as well

Exact hat algebra

```
Lhat
```

```
## 2-element Vector{Float64}:
##  0.6967741048725734
##  1.1299539550546118
```

Decreasing productivity in region 1 leads to reallocation of workers to region 2 as workers search for higher real wages

Armington with dynamic migration

Now let's introduce dynamics in the migration decision:

- Time $t = 0, \ldots, T$
- Same static goods market in each period t
- Each region j is populated with L_{jt} households where $\sum_{j \in N} L_{jt} = 1$
- Productivity in each time is A_{jt}
- Households are forward-looking and have perfect information
- Households discount the future at $\beta \in (0, 1)$
- Moving from i to j has a multiplicative utility cost $\mu_{ij} \in (0, 1]$
- Households work and consume at the beginning of the period, migrate at the end of the period

Armington with dynamic migration

We can write the household's objective as:

$$v_{jt} = \max_{i \in N} \log \frac{w_{jt}}{P_{jt}} + \beta \mathbb{E}[v_{it+1}] - \mu_{ji} + \varepsilon_{it}$$

where the idiosyncratic shock is destination-specific

Armington with dynamic migration

We can write the household's objective as:

$$v_{jt} = \max_{i \in N} \log \frac{w_{jt}}{P_{jt}} + \beta \mathbb{E}[v_{it+1}] - \mu_{ji} + \varepsilon_{it}$$

where the idiosyncratic shock is destination-specific

The share of households migrating from j to i at time t is:

$$\pi_{jit} = \frac{\exp\left(\beta \mathbb{E}[v_{it+1}] - \mu_{ji}\right)}{\sum_{k \in N} \exp\left(\beta \mathbb{E}[v_{kt+1}] - \mu_{jk}\right)}$$

The share of households in j at time t is still L_j

Armington with dynamic migration: labor supply

$$\pi_{jit} = \frac{\exp\left(\beta \mathbb{E}[v_{it+1}] - \mu_{ji}\right)}{\sum_{k \in N} \exp\left(\beta \mathbb{E}[v_{kt+1}] - \mu_{jk}\right)}$$

We now have our **dynamic** labor supply equation which depends on expected future payoffs and migration costs

Dynamic hat algebra

Armington: dynamic hat algebra

Now that our problem is dynamic we need to make one additional notational tweak: **dots/time changes**

$$\dot{Z}_{jt+1} \equiv Z_{jt+1}/Z_{jt}$$

The dot version of a variable is the relative time change between two periods

Armington: dynamic hat algebra

Now that our problem is dynamic we need to make one additional notational tweak: **dots/time changes**

$$\dot{Z}_{jt+1} \equiv Z_{jt+1}/Z_{jt}$$

The dot version of a variable is the relative time change between two periods

Then, the **dynamic hat** variable is the counterfactual relative to the factual in time changes:

$$\hat{Z}_{jt+1} \equiv \dot{Z}'_{jt+1}/\dot{Z}_{jt+1} = \frac{Z'_{jt+1}/Z'_{jt}}{Z_{jt+1}/Z_{jt}}$$

Armington: dynamic hat algebra

$$\hat{Z}_{jt+1} \equiv \dot{Z}'_{jt+1} / \dot{Z}_{jt+1} = \frac{Z'_{jt+1}/Z'_{jt}}{Z_{jt+1}/Z_{jt}}$$

In the static model using hat variables let us get around knowing the levels of most exogenous variables

Armington: dynamic hat algebra

$$\hat{Z}_{jt+1} \equiv \dot{Z}'_{jt+1}/\dot{Z}_{jt+1} = \frac{Z'_{jt+1}/Z'_{jt}}{Z_{jt+1}/Z_{jt}}$$

In the static model using hat variables let us get around knowing the levels of most exogenous variables

In the dynamic model using dynamic hat variables will let us get around knowing the levels of time-varying exogenous (common) variables: this is like a structural difference-in-differences

Armington: dynamic hat algebra

$$\hat{Z}_{jt+1} \equiv \dot{Z}'_{jt+1}/\dot{Z}_{jt+1} = \frac{Z'_{jt+1}/Z'_{jt}}{Z_{jt+1}/Z_{jt}}$$

In the static model using hat variables let us get around knowing the levels of most exogenous variables

In the dynamic model using dynamic hat variables will let us get around knowing the levels of time-varying exogenous (common) variables: this is like a structural difference-in-differences

Lets put our equilibrium conditions in dynamic hat notation starting with the labor supply equation

Armington: dynamic hat algebra

$$\pi_{jit} = \frac{\exp\left(\beta \mathbb{E}[v_{it+1}] - \mu_{ji}\right)}{\sum_{k \in N} \exp\left(\beta \mathbb{E}[v_{kt+1}] - \mu_{jk}\right)}$$

$$\dot{\pi}_{jit+1} = \frac{\frac{\exp\left(\beta \mathbb{E}[v_{it+2}] - \mu_{ji}\right)}{\exp\left(\beta \mathbb{E}[v_{it+1}] - \mu_{ji}\right)}}{\frac{\sum_{k \in N} \exp\left(\beta \mathbb{E}[v_{kt+2}] - \mu_{jk}\right)}{\sum_{l \in N} \exp\left(\beta \mathbb{E}[v_{lt+1}] - \mu_{jl}\right)}}$$

Next, let $u_{it} \equiv \exp(\mathbb{E}\left[v_{it}\right])$ to keep notation simple later, and use the multiply and divide trick to put into dot terms

Armington: dynamic hat algebra

$$\dot{\pi}_{jit+1} = \frac{\frac{\exp\left(\beta\mathbb{E}[v_{it+2}]-\mu_{ji}\right)}{\exp\left(\beta\mathbb{E}[v_{it+1}]-\mu_{ji}\right)}}{\frac{\sum_{k\in N}\exp\left(\beta\mathbb{E}[v_{kt+2}]-\mu_{jk}\right)}{\sum_{l\in N}\exp\left(\beta\mathbb{E}[v_{lt+1}]-\mu_{jl}\right)}}$$

$$\dot{\pi}_{jit+1} = \frac{\dot{u}_{it+2}^{\beta}}{\frac{\sum_{k\in N}\exp\left(\beta\mathbb{E}[v_{kt+2}]-\mu_{jk}\right)}{\sum_{l\in N}\exp\left(\beta\mathbb{E}[v_{lt+1}]-\mu_{jl}\right)}} = \frac{\dot{u}_{it+2}^{\beta}}{\frac{\sum_{k\in N}\exp\left(\beta\mathbb{E}[v_{kt+2}]-\mu_{jk}\right)\frac{\exp\left(\beta\mathbb{E}[v_{kt+1}]-\mu_{jk}\right)}{\exp\left(\beta\mathbb{E}[v_{kt+1}]-\mu_{jk}\right)}}{\sum_{l\in N}\exp\left(\beta\mathbb{E}[v_{lt+1}]-\mu_{jl}\right)}}$$

$$\dot{\pi}_{jit+1} = \frac{\dot{u}_{it+2}^{\beta}}{\sum_{k\in N}\dot{u}_{kt+2}^{\beta}\frac{\exp\left(\beta\mathbb{E}[v_{kt+2}]-\mu_{jk}\right)}{\sum_{l\in N}\exp\left(\beta\mathbb{E}[v_{lt+1}]-\mu_{jl}\right)}} = \frac{\dot{u}_{it+2}^{\beta}}{\sum_{k\in N}\pi_{jkt}\dot{u}_{kt+2}^{\beta}}$$

Armington: dynamic hat algebra

$$\dot{\pi}_{jit+1} = \frac{\dot{u}_{it+2}^{\beta}}{\sum_{k \in N} \pi_{jkt} \dot{u}_{kt+2}^{\beta}}$$

By putting migration into time changes, we differenced out time-invariant migration costs

Armington: dynamic hat algebra

$$\dot{\pi}_{jit+1} = \frac{\dot{u}_{it+2}^{\beta}}{\sum_{k \in N} \pi_{jkt} \dot{u}_{kt+2}^{\beta}}$$

By putting migration into time changes, we differenced out time-invariant migration costs

But now we have time changes in another endogenous variable $u_{it+2} = \exp(\mathbb{E}[v_{it+2}])$ so we need another equilibrium condition (in time changes)

Armington: dynamic hat algebra

We will use the T1EV version of the Bellman:

$$\mathbb{E}[v_{jt}] = \log u_{jt} = \log \frac{w_{jt}}{P_{jt}} + \log\left(\sum_{i \in N} \exp\left(\beta v_{it+1} - \mu_{ji}\right)\right)$$

Armington: dynamic hat algebra

We will use the T1EV version of the Bellman:

$$\mathbb{E}[v_{jt}] = \log u_{jt} = \log \frac{w_{jt}}{P_{jt}} + \log\left(\sum_{i \in N} \exp\left(\beta v_{it+1} - \mu_{ji}\right)\right)$$

Exponentiate both sides and then take time differences:

$$\dot{u}_{jt+1} = \frac{\dot{w}_{jt+1}}{\dot{P}_{jt+1}} \frac{\sum_{i \in N} \exp\left(\beta v_{it+2} - \mu_{ji}\right)}{\sum_{l \in N} \exp\left(\beta v_{lt+1} - \mu_{jl}\right)}$$

Next use the multiply and divide trick

Armington: dynamic hat algebra

$$\dot{u}_{jt+1} = \frac{\dot{w}_{jt+1}}{\dot{P}_{jt+1}} \frac{\sum_{i \in N} \exp\left(\beta v_{it+2} - \mu_{ji}\right) \frac{\exp\left(\beta v_{it+1} - \mu_{ji}\right)}{\exp\left(\beta v_{it+1} - \mu_{ji}\right)}}{\sum_{l \in N} \exp\left(\beta v_{lt+1} - \mu_{jl}\right)}$$

$$= \frac{\dot{w}_{jt+1}}{\dot{P}_{jt+1}} \sum_{i \in N} \frac{\exp\left(\beta v_{it+2} - \mu_{ji}\right)}{\exp\left(\beta v_{it+1} - \mu_{ji}\right)} \frac{\exp\left(\beta v_{it+1} - \mu_{ji}\right)}{\sum_{l \in N} \exp\left(\beta v_{lt+1} - \mu_{jl}\right)}$$

$$\dot{u}_{jt+1} = \frac{\dot{w}_{jt+1}}{\dot{P}_{jt+1}} \sum_{i \in N} \dot{u}_{it+2}^{\beta} \pi_{jit}$$

Now we have $\dot{u}$ as a function of itself and other dot variables

Armington: dynamic hat algebra

Next, do the same for the market clearing condition

$$w_{jt}L_{jt} = \sum_{i\in N} \frac{a_{ij}\left(\frac{\tau_{ij}w_{it}}{A_{it}}\right)^{1-\sigma}}{\sum_{k\in N} a_{kj}\left(\frac{\tau_{kj}w_{kt}}{A_{kt}}\right)^{1-\sigma}} w_{it}L_{it}$$

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You can prove to yourself that it is:

$$\dot{w}_{jt+1}\dot{L}_{jt+1} = \sum_{i \in N} \frac{\frac{X_{jit}}{w_{jt}L_{jt}}\left(\frac{\dot{w}_{jt+1}}{\dot{A}_{jt+1}}\right)^{1-\sigma} \dot{w}_{it+1}\dot{L}_{it+1}}{\sum_{k \in N} \lambda_{kit}\left(\frac{\dot{w}_{kt+1}}{\dot{A}_{kt+1}}\right)^{1-\sigma}}$$

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We have our three equilibrium conditions in time changes

Along with the labor transition $L_{jt+1} = \sum_{i \in N} \pi_{ijt} L_{it}$, and changes in prices that are easy to solve for $\dot{P}_{jt+1}$, we can then solve for the dynamic equilibrium of the economy given some sequence of **changes** in productivity $\dot{A}_{jt+1}$

Armington: dynamic hat algebra

$$\dot{w}_{jt+1}\dot{L}_{jt+1} = \sum_{i\in N} \frac{\frac{X_{jit}}{w_{jt}L_{jt}}\left(\frac{\dot{w}_{jt+1}}{\dot{A}_{jt+1}}\right)^{1-\sigma}\dot{w}_{it+1}\dot{L}_{it+1}}{\sum_{k\in N}\lambda_{kit}\left(\frac{\dot{w}_{kt+1}}{\dot{A}_{kt+1}}\right)^{1-\sigma}}$$

$$\dot{\pi}_{jit+1} = \frac{\dot{u}_{it+2}^{\beta}}{\sum_{k\in N}\pi_{jkt}\dot{u}_{kt+2}^{\beta}} \qquad \dot{u}_{jt+1} = \frac{\dot{w}_{jt+1}}{\dot{P}_{jt+1}}\sum_{i\in N}\dot{u}_{it+2}^{\beta}\pi_{jit}$$

$$L_{jt+1} = \sum_{i\in N}\pi_{ijt}L_{it} \qquad \dot{P}_{jt+1} = \left(\sum_{k\in N}\left(\frac{\dot{w}_{kt+1}}{\dot{A}_{kt+1}}\right)^{1-\sigma}\lambda_{kjt}\right)^{\frac{1}{1-\sigma}}$$

We also impose $\dot{u}_{it}$ converges to 1 (else can't solve the problem)

Armington: dynamic hat algebra

How do we solve it?

Armington: dynamic hat algebra

How do we solve it?

We essentially have two nested problems:

1. A static market-clearing problem at each time t (conditional on the labor allocation)
2. A dynamic migration problem (conditional on the sequence of wages and prices)

Armington: dynamic hat algebra

Pseudocode might look like this:

```
while error > tolerance (outer loop)
  compute sequence of migration shares given initial conditions and expected values
  compute sequence of labor given initial conditions and migration shares
  for each time t (inner loop)
      solve for wages and prices that clear the goods market
  end
  compute the sequence of expected values
  compute error in expected values since last iteration
end
```

`solve_arm_dyn_mig(X, lambda, w, L, pi, Ahat, sigma, beta, tol, damp)`