Lecture 01

Intro to computing

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AEM 7130

Software and stuff

Necessary things to download to follow along today and in the future:

- Git
- Julia or JuliaPro
- VSCode
- A GitHub account

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For this lecture you will need the following Julia packages

```
import Pkg; Pkg.add("ForwardDiff"); Pkg.add("Distributions"); Pkg.add("BenchmarkTools")
using ForwardDiff, Distributions, BenchmarkTools
```

What this class is about

1. Learning how to compute dynamic and spatial models
2. Other useful computational techniques and details

What you need to succeed in this course

1. ECON 6090 and ECON 6170
2. ECON 6130 or AEM 7040
3. Previous coding experience or willingness to spend some time learning as you go

Course materials

1. Everything we use in the course will be **freely available** and posted to the course GitHub (details next week on how to use Git)
2. Books (free from the library or authors' websites):
    1. Judd (1998)
    2. Miranda and Fackler (2002)
    3. Nocedal and Wright (2006)

Things to do before next class

- Spend some time reading up on Julia if you don't use it already:
  - Learning Julia
  - QuantEcon Julia lectures

What we will cover in the class

1. Basic computing, arithmetic, calculus, and linear algebra on a computer
2. Shell and Julia coding, version control, reproducibility, workflow
3. Optimization
4. Solving dynamic models
5. Solving spatial models

What you have to do

- Come to class
- 4 computational problem sets
- Final research project proposal
- Final research project
- One presentation of a paper from the literature

Important days / times

- Office hours: Tuesday 3:00-4:00
- Final project proposal: March 17
- Final project paper: May 19

Grading

- Problem sets: 40% (10% each)
- Final project proposal: 15%
- Final project paper/presentation: 25%
- Class participation: 10%
- Computational paper presentation: 10%

Problem sets (10% each)

You **must** use Julia

- Code must be written in `.jl` scripts
- You must use Julia project management tools, e.g. `Pkg.generate()`, `Pkg.activate()`, `Pkg.instantiate()`
- Everything must be nested in a wrapper function
- It must work just by running the wrapper function

You can work in groups of 2

Problem sets will be where you **implement** the techniques we learn in class on your own, but we will be doing our fair share of coding in class

Problem sets (10% each)

Why am I making you do problem sets this way?

If you want to publish in an AEA journal (amongst others now..) you need to have good practices, other journals are following suit

Julia is very good for reproducibility

Computational paper presentations (10%)

Everyone will present a paper or package starting in a few weeks

The paper can apply methods we've learned about (or will learn about), or can be a new method that we have not covered

The package must be something related to the methods we are learning

You must consult with me at least 1 week prior to your scheduled presentation date to ensure the paper/package is appropriate for a presentation

The syllabus has some pre-approved papers you can choose from under the **Applications** header

Final project (25% paper, 15% proposal)

The final project will be the beginning of a **computationally-driven** research project or an extension of an existing paper with new methods

Proposals will be due about half way through the class

Final project (25% paper, 15% proposal)

The only requirement is that the project **cannot be computationally trivial** (i.e. no applied micro papers)

It can be numerical, empirical, whatever

Everyone will present their final projects in the last week of class

More details on the syllabus and to come later

Why computational methods?

Why do we need computational methods?

1st year PhD core: everything is typically analytically tractable

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Including OLS: $\hat{\beta} = (X'X)^{-1}X'Y$

Not all economic models have closed-form solutions, and others can't have closed-form solutions with losing important economic content

This is generally true for dynamic and spatial models

What can we compute?

We can use computation + theory to answer **quantitative** questions

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Theory can't give us welfare in dollar terms

Theory can't tell us the value of economic primitives

What can we compute?

Theory often relies on strong assumptions like:

What can we compute?

Theory often relies on strong assumptions like:

- log utility (lose income vs substitution)
- zero frictions
- strictly linear transitions (natural phenomena don't follow this)
- static decisionmaking

It can be unclear what the cost of these assumptions are

Example 1

Suppose we have a constant elasticity demand function: $q(p) = p^{-0.2}$

In equilibrium, quantity demanded is $q^* = 2$

What price clears the market in equilibrium?

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What price clears the market in equilibrium?

Just invert the demand function:

$2 = p^{-0.2}$

$p^* = 2^{-5}$✓

Your calculator can do the rest

Example 2

Suppose the demand function is now: $q(p) = 0.5p^{-0.2} + 0.5p^{-0.5}$, a weighted average of two CE demand functions

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Suppose the demand function is now: $q(p) = 0.5p^{-0.2} + 0.5p^{-0.5}$, a weighted average of two CE demand functions

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First, does a solution exist?

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In equilibrium, quantity demanded is $q^* = 2$

What price clears the market in equilibrium?

First, does a solution exist?

Yes, why?

Example 2

$q(p)$ is monotonically decreasing

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$q(p)$ is greater than 2 at $p = 0.1$ and less than 2 at $p = 0.2$

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$q(p)$ is monotonically decreasing

$q(p)$ is greater than 2 at $p = 0.1$ and less than 2 at $p = 0.2$

$\rightarrow$ by intermediate value theorem $q(p) = 2$ somewhere in $(0.1, 0.2)$

Example 2

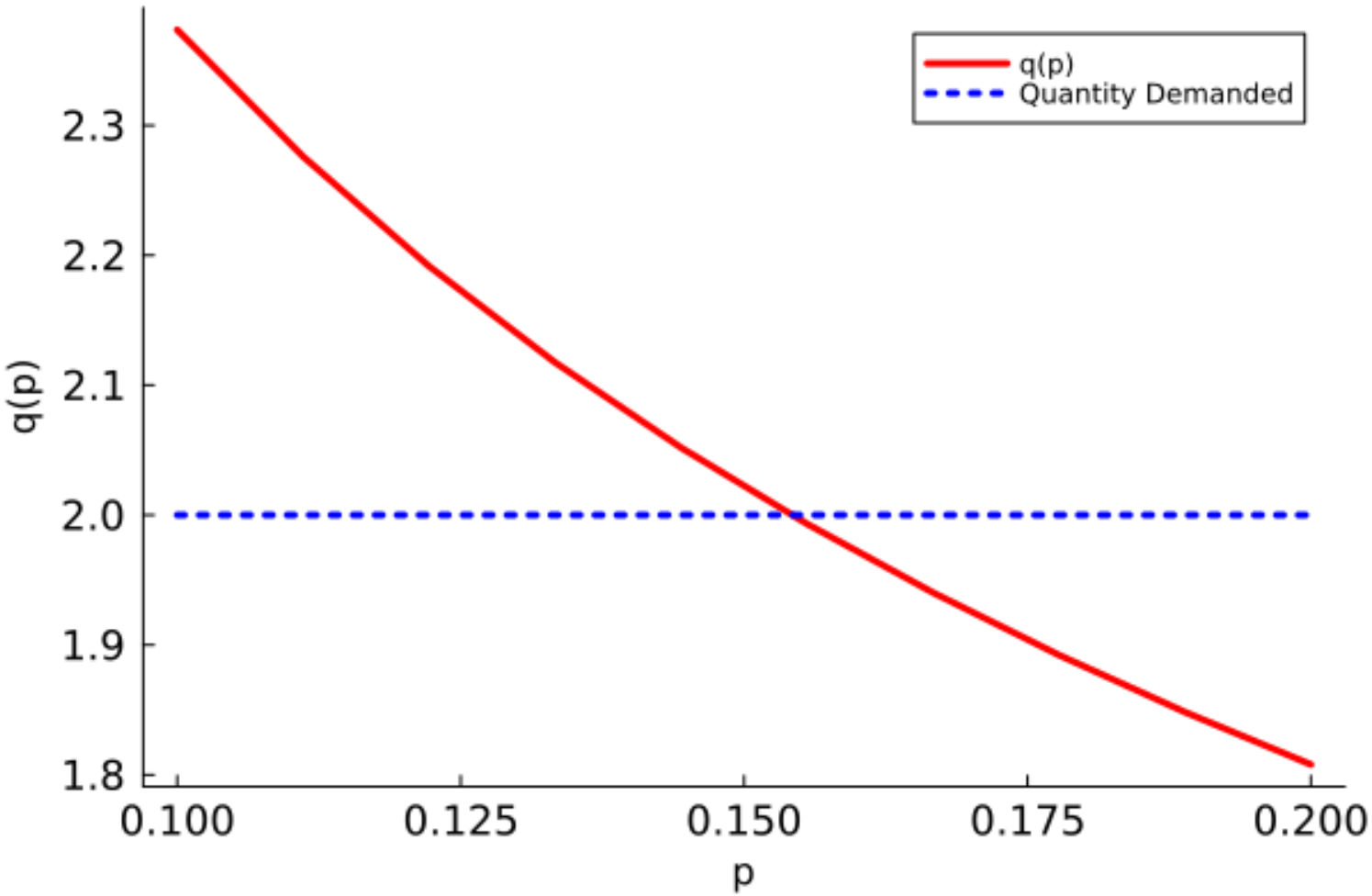
```
# We know solution is between .1 and .2
x = collect(range(.1, stop = .2, length = 10)) # generate evenly spaced grid
q_d = ones(size(x)).*2                         # generate equal length vector of qd=2

# Price function
price(p) = p.^(-0.2)/2 .+ p.^(-0.5)/2

# Get corresponding quantity values at these prices
y = price(x)
```

Now plot q_d and $q(p)$

Example 2


q(p)
Quantity Demanded
2.3
2.2
2.1
2.0
1.9
1.8
q(p)
0.100
0.125
0.150
0.175
0.200
p

Example 2

Notice: if we let $t = p^{-0.1}$ then:

$q(t) = 0.5t^2 + 0.5t^5$

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Notice: if we let $t = p^{-0.1}$ then:

$q(t) = 0.5t^2 + 0.5t^5$

Can we solve for t now?

No! Closed-form solutions to fifth order polynomials are not guaranteed to exist!

So how do we solve the problem?

Newton's method

Iteratively do the following:

1. Guess solution to: $q(p) - q^* = 0 \rightarrow q(p) - 2 = 0$
2. Approximate the function with local second order polynomial around guess
3. Solve this easier equation
4. Solution is the new guess
5. Stop if previous guess and new guess are sufficiently close

We will learn more about this and why it works in a later class

Newton code

```
# Define demand functions
demand(p) = p^(-0.2)/2 + p^(-0.5)/2 - 2     # quantity minus price
demand_grad(p) = .1*p^(-1.2) + .25*p^(-1.5) # demand gradient
```

Newton code

```
# Define demand functions
demand(p) = p^(-0.2)/2 + p^(-0.5)/2 - 2     # quantity minus price
demand_grad(p) = .1*p^(-1.2) + .25*p^(-1.5) # demand gradient
```

```
function find_root_newton(demand, demand_grad)
    p = .3        # initial guess
    deltap = 1e10 # initialize stepsize

    while abs(deltap) > 1e-4
        deltap = demand(p)/demand_grad(p)
        p += deltap
        println("Intermediate guess of p = $(round(p,digits=3)).")
    end
    println("The solution is p = $(round(p,digits=3)).")
    return p
end;
```

Newton code

```
# Solve for price
find_root_newton(demand, demand_grad)
```

```
## Intermediate guess of p = 0.068.
## Intermediate guess of p = 0.115.
## Intermediate guess of p = 0.147.
## Intermediate guess of p = 0.154.
## Intermediate guess of p = 0.154.
## Intermediate guess of p = 0.154.
## The solution is p = 0.154.

## 0.15419764093200633
```

Example 3

Consider a two period ag commodity market model

Period 1: Farmer makes acreage decisions for planting

Period 2: Per-acre yield realizes, equilibrium crop price clears the market

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The farmer's policy function is: $a(E[p]) = \frac{1}{2} + \frac{1}{2}E[p]$

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Consider a two period ag commodity market model

Period 1: Farmer makes acreage decisions for planting
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After planting, yield $\hat{y}$ realizes, producing a total quantity $q = a\hat{y}$ of the crop

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After planting, yield $\hat{y}$ realizes, producing a total quantity $q = a\hat{y}$ of the crop

Demand is given by $p(q) = 3 - 2q$

Yield is given by $\hat{y} \sim \mathcal{N}(1, 0.1)$

How much acreage gets planted?

$$p(\hat{y}) = 3 - 2a\hat{y}$$

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$$a = \frac{1}{2} + \frac{1}{2}(3 - 2aE[\hat{y}])$$

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$$p(\hat{y}) = 3 - 2a\hat{y}$$

$$a = \frac{1}{2} + \frac{1}{2}(3 - 2aE[\hat{y}])$$

Rearrange and solve:

$$a^* = 1$$

How much acreage gets planted?

$p(\hat{y}) = 3 - 2a\hat{y}$

$a = \frac{1}{2} + \frac{1}{2}(3 - 2aE[\hat{y}])$

Rearrange and solve:

$a^* = 1$

Now suppose the government implements a price floor on the crop of $p > 1$ so we have that $p(\hat{y}) = \max(1, 3 - 2a\hat{y})$

How much acreage does the farmer plant?

How much acreage gets planted?

This is analytically intractable

How much acreage gets planted?

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The max operator is non-linear so we can't pass the expectation through

$$E[\max(1, 3 - 2a\hat{y})] \neq \max(1, E[3 - 2a\hat{y}])$$

How much acreage gets planted?

This is analytically intractable

The max operator is non-linear so we can't pass the expectation through

$E[\max(1, 3 - 2a\hat{y})] \neq \max(1, E[3 - 2a\hat{y}])$

$\rightarrow$ we need to solve this numerically

Function iteration

We can solve this using another technique called **function iteration**

```
# Function iteration method to find a root
function find_root_fi(mn, variance)

    y = randn(1000)*sqrt(variance) .+ mn # draws of the random variable
    a = 1.                               # initial guess
    differ = 100.                        # initialize error
    exp_price = 1.                       # initialize expected price

    while differ > 1e-4
        a_old = a                        # save old acreage
        p = max.(1, 3 .- 2 .*a.*y)       # compute price at all distribution points
        exp_price = mean(p)              # compute expected price
        a = 1/2 + 1/2*exp_price          # get new acreage planted given new price
        differ= abs(a - a_old)           # change in acreage planted
        println("Intermediate acreage guess: $(round(a,digits=3))")
    end

    return a, exp_price
end
```

Function iteration

```
acreage, expected_price = find_root_fi(1, 0.1);
println("The optimal number of acres to plant is $(round(acreage, digits = 3)).\nThe expected pr
```

```
## The optimal number of acres to plant is 1.094.
## The expected price is 1.188.
```

Quantifying speed and accuracy

Big O notation

How do we quantify speed and accuracy of computational algorithms?

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Big O notation describes the limiting behavior of a function when the argument tends towards a particular value or infinity

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Programming context: Describes the limiting behavior of algorithms in terms of run time/memory/accuracy as input size grows

Big O notation

How do we quantify speed and accuracy of computational algorithms?

Big O notation describes the limiting behavior of a function when the argument tends towards a particular value or infinity

Programming context: Describes the limiting behavior of algorithms in terms of run time/memory/accuracy as input size grows

You've seen this before in the expression of Taylor series' errors

Big O Notation

Written as: **O(F(x))**

Here is how to think about it:

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Here is how to think about it:

O(x): linear

- Time to solve increases linearly in size of input x
- Accuracy changes linearly in size of input x

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Examples?

Big O Notation

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O(x): linear

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Examples?

Time to find a particular (e.g. maximum) value in an unsorted array

→ For each element, check whether it is the value we want

Big O Notation

$O(c^x)$: exponential

- Time to solve increases exponentially in input x
- Accuracy changes exponentially in input x

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Examples?

Big O Notation

$O(c^x)$: exponential

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- Accuracy changes exponentially in input x

Examples?

Time to solve a standard dynamic program, ex traveling salesman

$\rightarrow$ For each city $i = 1, \ldots, n$, solve a Bellman as a function of all other cities

Big O Notation

$\mathbf{O(n!)}$: factorial

- Time to solve increases factorially in input x
- Accuracy changes factorially in input x

Big O Notation

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Examples?

Big O Notation

$\mathbf{O(n!)}$: factorial

- Time to solve increases factorially in input x
- Accuracy changes factorially in input x

Examples?

Solving traveling salesman by brute force

$\rightarrow$ Obtain travel time for all possible combinations of intermediate cities

Big O Notation: Accuracy example

This is how you have probably seen Big O used before:

Big O Notation: Accuracy example

This is how you have probably seen Big O used before:

Taylor series for $sin(x)$ around zero:

$$sin(x) \approx x - x^3/3! + x^5/5! + O(x^7)$$

What does $O(x^7)$ mean here?

Big O Notation: Accuracy example

$$sin(x) \approx x - x^3/3! + x^5/5! + O(x^7)$$

Big O Notation: Accuracy example

$sin(x) \approx x - x^3/3! + x^5/5! + O(x^7)$

As we move away from 0 to some x, the upper bound of the growth rate in the error of our approximation to $sin(x)$ is x^7

We are approximating about zero so x is small and x^n is decreasing in n

Big O Notation: Accuracy example

$sin(x) \approx x - x^3/3! + x^5/5! + O(x^7)$

As we move away from 0 to some x, the upper bound of the growth rate in the error of our approximation to $sin(x)$ is x^7

We are approximating about zero so x is small and x^n is decreasing in n

For small x, higher order polynomials mean the error will grow slower and we have a better local approximation

Taylor expansions

```
# fifth and third order Taylor approximations
sin_error_5(x) = sin(x) - (x - x^3/6 + x^5/120)
sin_error_3(x) = sin(x) - (x - x^3/6)
```

Taylor expansions

```
# fifth and third order Taylor approximations
sin_error_5(x) = sin(x) - (x - x^3/6 + x^5/120)
sin_error_3(x) = sin(x) - (x - x^3/6)
```

```
println("Error of fifth-order approximation at x = .001 is: $(sin_error_5(.001))
Error of third-order approximation at x = .001 is: $(sin_error_3(.001))
Error of fifth-order approximation at x = .01 is: $(sin_error_5(.01))
Error of third-order approximation at x = .01 is: $(sin_error_3(.01))
Error of fifth-order approximation at x = .1 is: $(sin_error_5(.1))
Error of third-order approximation at x = .1 is: $(sin_error_3(.1))")
```

```
## Error of fifth-order approximation at x = .001 is: 0.0
## Error of third-order approximation at x = .001 is: 8.239936510889834e-18
## Error of fifth-order approximation at x = .01 is: -1.734723475976807e-18
## Error of third-order approximation at x = .01 is: 8.333316675601665e-13
## Error of fifth-order approximation at x = .1 is: -1.983851971587569e-11
## Error of third-order approximation at x = .1 is: 8.331349481138783e-8
```

Big O Notation: Speed examples

Here are a few examples for fundamental computational methods

Big O Notation: O(1)

O(1): algorithm executes in **constant time**

The size of the input does not affect execution speed

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Big O Notation: O(1)

O(1): algorithm executes in **constant time**

The size of the input does not affect execution speed

accessing a specific location in an array

Big O Notation: O(x)

O(x): algorithm executes in **linear time**

Execution speed grows linearly in input size

Example:

Big O Notation: O(x)

O(x): algorithm executes in **linear time**

Execution speed grows linearly in input size

Example:

inserting an element into an arbitrary location in a 1 dimensional array

Bigger array $\rightarrow$ need to shift around more elements in memory to accommodate the new element

Big O Notation: $O(x^2)$

$\mathbf{O(x^2)}$ **:** algorithm executes in **quadratic time**

More generally called polynomial time for x^n

Execution speed grows quadratically in input size

Example:

Big O Notation: $O(x^2)$

$\mathbf{O(x^2)}$ **:** algorithm executes in **quadratic time**

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Example:

bubble sort, step through a list, compare adjacent elements, swap if in the wrong order

Big O Notation: $O(x^2)$

$\mathbf{O(x^2)}$ **:** algorithm executes in **quadratic time**

More generally called polynomial time for x^n

Execution speed grows quadratically in input size

Example:

bubble sort, step through a list, compare adjacent elements, swap if in the wrong order

matrix inversion, most algorithms (e.g. LU decomposition) solve in polynomial time

Computer arithmetic

Computer arithmetic - storage

Question: which numbers can be represented by a computer?

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Before the answer, how are numbers physically represented by a computer?

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Binary: a base 2 number system

Each digit can only take on 0 or 1

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Before the answer, how are numbers physically represented by a computer?

Binary: a base 2 number system

Each digit can only take on 0 or 1

Base 10: each digit can take on 0-9

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This imposes a strict limitation on the storage of numbers

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Answer: a subset of the rational numbers

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This imposes a strict limitation on the storage of numbers

Numbers are stored as: $\pm m b^{\pm n}$

m is the mantissa/significand, b is the base, n is the exponent

All three are integers

Computer arithmetic - storage

$\pm m b^{\pm n}$

The significand typically gives the significant digits

The exponent scales the number up or down in magnitude

Computer arithmetic - storage

The size of numbers a computer can represent is limited
by how much space is typically allocated for a real number

Computer arithmetic - storage

The size of numbers a computer can represent is limited
by how much space is typically allocated for a real number

Space allocations are usually 64 bits: 53 for m and 11 for n

```
println(typeof(5.0))
```

```
## Float64
```

```
println(typeof(5))
```

```
## Int64
```

Computer arithmetic - storage

`Int64` means it is a **integer** with 64 bits of storage

`Float64` means it is a **floating point number** with 64 bits of storage

Floating point just means $b^{\pm n}$ can move the decimal point around in the significand

`Int64` and `Float64` are different, this will be important later

The limits of computers

Limitations on storage suggest three facts

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1. There exists a **machine epsilon** which denotes the smallest relative quantity representible by a computer

The limits of computers

Limitations on storage suggest three facts

1. There exists a **machine epsilon** which denotes the smallest relative quantity representible by a computer

Machine epsilon is the smallest ϵ such that a machine can always distinguish

$$N + \epsilon > N > N - \epsilon$$

The limits of computers

```
println("Machine epsilon ϵ is $(eps(Float64))")
```

```
## Machine epsilon ϵ is 2.220446049250313e-16
```

```
println("Is 1 + ϵ/2 > 1? $(1 + eps(Float64)/2 > 1)")
```

```
## Is 1 + ϵ/2 > 1? false
```

```
println("Is 1 - ϵ/2 < 1? $(1 - eps(Float64)/2 < 1)")
```

```
## Is 1 - ϵ/2 < 1? true
```

The limits of computers

```
println("The smallest representable number larger than 1.0 is $(nextfloat(1.0))")
```

```
## The smallest representable number larger than 1.0 is 1.0000000000000002
```

```
println("The largest representable number smaller than 1.0 is $(prevfloat(1.0))")
```

```
## The largest representable number smaller than 1.0 is 0.9999999999999999
```

The limits of computers

Machine epsilon changes depending on the amount of storage allocated

The limits of computers

Machine epsilon changes depending on the amount of storage allocated

```
println("32 bit machine epsilon is $(eps(Float32))")
```

```
## 32 bit machine epsilon is 1.1920929e-7
```

```
println("Is 1 + ϵ/2 > 1? $(Float32(1) + eps(Float32)/2 > 1)")
```

```
## Is 1 + ϵ/2 > 1? false
```

```
println("Is 1 - ϵ/2 < 1? $(Float32(1) - eps(Float32)/2 < 1)")
```

```
## Is 1 - ϵ/2 < 1? true
```

The limits of computers

Machine epsilon changes depending on the amount of storage allocated

```
println("32 bit machine epsilon is $(eps(Float32))")
```

```
## 32 bit machine epsilon is 1.1920929e-7
```

```
println("Is 1 + ϵ/2 > 1? $(Float32(1) + eps(Float32)/2 > 1)")
```

```
## Is 1 + ϵ/2 > 1? false
```

```
println("Is 1 - ϵ/2 < 1? $(Float32(1) - eps(Float32)/2 < 1)")
```

```
## Is 1 - ϵ/2 < 1? true
```

Theres a tradeoff between precision and storage requirements

The limits of computers

2. There is a smallest representable number

```
println("64 bit smallest float is $(floatmin(Float64))")
```

```
## 64 bit smallest float is 2.2250738585072014e-308
```

```
println("32 bit smallest float is $(floatmin(Float32))")
```

```
## 32 bit smallest float is 1.1754944e-38
```

```
println("16 bit smallest float is $(floatmin(Float16))")
```

```
## 16 bit smallest float is 6.104e-5
```

The limits of computers

3. There is a largest representable number

```
println("64 bit largest float is $(floatmax(Float64))")
```

```
## 64 bit largest float is 1.7976931348623157e308
```

```
println("32 bit largest float is $(floatmax(Float32))")
```

```
## 32 bit largest float is 3.4028235e38
```

```
println("16 bit largest float is $(floatmax(Float16))")
```

```
## 16 bit largest float is 6.55e4
```

The limits of computers

```
println("The largest 64 bit integer is $(typemax(Int64))")
```

```
## The largest 64 bit integer is 9223372036854775807
```

```
println("Add one to it and we get: $(typemax(Int64)+1)")
```

```
## Add one to it and we get: -9223372036854775808
```

```
println("It loops us around the number line: $(typemin(Int64))")
```

```
## It loops us around the number line: -9223372036854775808
```

The limits of computers

The scale of your problem matters

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If a parameter or variable is > floatmax or < floatmin, you will have a very bad time

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Scale numbers appropriately (e.g. millions of dollars, not millionths of cents)

Computer arithmetic: Error

We can only represent a finite number of numbers

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Error comes in two major and related forms:

1. Rounding
2. Truncation

Rounding

We will always need to round numbers to the nearest computer representable number, this introduces error

```
println("Half of π is: $(π/2)")
```

```
## Half of π is: 1.5707963267948966
```

Rounding

We will always need to round numbers to the nearest computer representable number, this introduces error

```
println("Half of π is: $(π/2)")
```

```
## Half of π is: 1.5707963267948966
```

The computer gave us a rational number, but $\pi/2$ should be irrational

Truncation

Lots of important numbers are defined by infinite sums $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

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Lots of important numbers are defined by infinite sums $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

It turns out that computers can't add up infinitely many terms because there is finite space

$\rightarrow$ we need to truncate the sum

Why does this matter?

Errors are small, who cares?

Why does this matter?

Errors are small, who cares?

You should!

Because errors can propagate and grow as you keep applying an algorithm (e.g. function iteration)

Error example 1

Consider a simple quadratic: $x^2 - 26x + 1 = 0$ with solution $x = 13 - \sqrt{168}$

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```
println("64 bit: 13 - √168 = $(13-sqrt(168))")
```

```
## 64 bit: 13 - √168 = 0.038518603184279 24
```

```
println("32 bit: 13 - √168 = $(convert(Float32,13-sqrt(168)))")
```

```
## 32 bit: 13 - √168 = 0.038518604
```

```
println("16 bit: 13 - √168 = $(convert(Float16,13-sqrt(168)))")
```

```
## 16 bit: 13 - √168 = 0.0385
```

Error example 2

Lets add and subtract some numbers and play around with the associative property of real numbers:

$x = (10^{-20} + 1) - 1$
$y = 10^{-20} + (1 - 1)$

Error example 2

Lets add and subtract some numbers and play around with the associative property of real numbers:

$$x = (10^{-20} + 1) - 1$$
$$y = 10^{-20} + (1 - 1)$$

Very clearly we should get $x = y$, but do we? Let's find out

Error example 2

```
x = (1e-20 + 1) - 1   # initialize x
y = 1e-20 + (1 - 1)   # initialize y
x_equals_y = (x == y) # store boolean of whether x == y
```

Error example 2

```
x = (1e-20 + 1) - 1   # initialize x
y = 1e-20 + (1 - 1)   # initialize y
x_equals_y = (x == y) # store boolean of whether x == y
```

```
if x_equals_y
    println("X equals Y!")
else
    println("X does not equal Y!")
    println("The difference is: $(x-y).")
end
```

```
## X does not equal Y!
## The difference is: -1.0e-20.
```

Error example 2

The two numbers were not equal, we got $y > x$

Why?

Error example 2

The two numbers were not equal, we got $y > x$

Why?

Adding numbers of greatly different magnitudesdoes not always work like you would want

Error example 2

```
x = (1e-20 + 1) - 1   # initialize x
y = 1e-20 + (1 - 1)   # initialize y
```

```
println("x is $x")
```

```
## x is 0.0
```

```
println("y is $y")
```

```
## y is 1.0e-20
```

Error example 2

```
x = (1e-20 + 1) - 1   # initialize x
y = 1e-20 + (1 - 1)   # initialize y
```

```
println("x is $x")
```

```
## x is 0.0
```

```
println("y is $y")
```

```
## y is 1.0e-20
```

When we added 10^{-20} to 1, it got rounded away

Error example 3

Lets just subtract two numbers: 100000.2 - 100000.1

We know the answer is: 0.1

Error example 3

Lets just subtract two numbers: 100000.2 - 100000.1

We know the answer is: 0.1

```
println("100000.2 - 100000.1 is: $(100000.2 - 100000.1)")
```

```
## 100000.2 - 100000.1 is: 0.09999999999126885
```

```
if (100000.2 - 100000.1) == 0.1
    println("and it is equal to 0.1")
else
    println("and it is not equal to 0.1")
end
```

```
## and it is not equal to 0.1
```

Error example 3

Why do we get this error?

Error example 3

Why do we get this error?

Neither of the two numbers can be precisely represented by the machine!

$$100000.1 \approx 8589935450993459 \times 2^{-33} = 100000.09999999997671693563461 30$$
$$100000.2 \approx 8589936309986918 \times 2^{-33} = 100000.19999999995343387126922 61$$

So their difference won't necessarily be 0.1

There are tools for approximate equality

```
isapprox(100000.2 - 100000.1, 0.1)
```

```
## true
```

Rounding and truncation recap

This matters, particularly when you're trying to evaluate logical expressions of equality

Calculus on a machine

Differentiation

Derivatives are important in economics for finding optimal allocations, etc

The formal definition of a derivative is:

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$$\frac{df(x)}{dx} = \lim_{t \to \infty} \frac{f(x+1/t) - f(x)}{1/t}$$

which we know a computer can't handle because of finite space to store t

Computer differentiation

How do we perform derivatives on computers if we can't take the limit?

Computer differentiation

How do we perform derivatives on computers if we can't take the limit?

Finite difference methods

Computer differentiation

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How do we perform derivatives on computers if we can't take the limit?

Finite difference methods

What does a finite difference approximation look like?

Forward difference

The forward difference looks exactly like the formal definition without the limit:

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$$\frac{df(x)}{dx} \approx \frac{f(x+h) - f(x)}{h}$$

Works the same for partial derivatives:

$$\frac{\partial g(x,y)}{\partial x} \approx \frac{g(x+h,y) - g(x,y)}{h}$$

Let's see how it works in practice by calculating derivatives of x^2 at $x = 2$

Forward difference

```
deriv_x_squared(h,x) = ((x+h)^2 - x^2)/h # derivative function
```

Forward difference

```
deriv_x_squared(h,x) = ((x+h)^2 - x^2)/h # derivative function
```

```
println("
        The deriviative with h=1e-8 is: $(deriv_x_squared(1e-8,2.))
        The deriviative with h=1e-12 is: $(deriv_x_squared(1e-12,2.))
        The deriviative with h=1e-30 is: $(deriv_x_squared(1e-30,2.))
        The deriviative with h=1e-1 is: $(deriv_x_squared(1e-1,2.))")
```

```
##
##          The deriviative with h=1e-8 is: 3.999999975690116
##          The deriviative with h=1e-12 is: 4.000355602329364
##          The deriviative with h=1e-30 is: 0.0
##          The deriviative with h=1e-1 is: 4.100000000000001
```

Error, it's there

None of the values we chose for h were perfect, but clearly some were better than others

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We want h to be as small as possible so thatwe can approximate the limit as well as we possibly can, BUT

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None of the values we chose for h were perfect, but clearly some were better than others

Why?

We face two opposing forces:

We want h to be as small as possible so thatwe can approximate the limit as well as we possibly can, BUT

If h is small then $f(x+h)$ is close to $f(x)$, we can run into rounding issues like we saw for $h = 10^{-30}$

Error, it's there

We can select h in an optimal fashion: $h = \max\{|x|, 1\}\sqrt{\epsilon}$

Error, it's there

We can select h in an optimal fashion: $h = \max\{|x|, 1\}\sqrt{\epsilon}$

There are proofs for why this is the case but generally testing out different h's works fine

How much error is in a finite difference?

Can we measure the error growth rate in h (i.e. Big O notation)?

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$$f(x+h) = f(x) + f'(x)h + O(h^2)$$

Recall $O(h^2)$ means the error in our approximation grows quadratically in h, we only did a linear approximation

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How can we use this to understand the error in our finite difference approximation?

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Rearrange to obtain: $f'(x) = \frac{f(x+h)-f(x)}{h} + O(h^2)/h$

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Forward differences have linear errors

If we halve h, we halve the error in our approximation (ignoring rounding/truncation issues)

Improvements on the forward difference

How can we improve the accuracy of the forward difference?

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The secant curve has the average slope of $f(x)$ on $[x, x+h]$

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First, **why** do we have error?

Because we are approximating the slope of a tangent curve at x
by a secant curve passing through $(x, x+h)$

The secant curve has the average slope of $f(x)$ on $[x, x+h]$

We want the derivative at x, which is on the edge of $[x, x+h]$, how about we **center** x?

Central differences

We can approximate $f'(x)$ in a slightly different way:

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$

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Is this an improvement on forward differences?

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How much error is in a central finite difference?

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + O(h^2)$$

Error falls quadratically in h, if we halve h we reduce error by 75%

Optimal selection of h for central differences is $h = \max\{|x|, 1\}\epsilon^{1/3}$

Why use anything but central differences?

Why would we ever use forward differences instead of central differences?

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For each central difference:

We need to compute $g(x_1 - h, x_2, \dots)$ and $g(x_1 + h, x_2, \dots)$ for each x_i

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For each central difference:

We need to compute $g(x_1 - h, x_2, \dots)$ and $g(x_1 + h, x_2, \dots)$ for each x_i

But for a forward difference we only need to compute $g(x_1, x_2, \dots)$ once and then $g(x_1 + h, x_2, \dots)$ for each x_i

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But for a forward difference we only need to compute $g(x_1, x_2, \dots)$ once and then $g(x_1 + h, x_2, \dots)$ for each x_i

Forward differences saves on # of operations at the expense of accuracy

Higher order finite differences

We can use these techniques to approximate higher order derivatives

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For example, take two third order Taylor expansions

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Add the two expressions and then divide by h^2 to get

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Add the two expressions and then divide by h^2 to get

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + O(h^2)$$

Differentiation without error?

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One way is to code up the actual derivative

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One way is to code up the actual derivative

```
deriv_x_squared(x) = 2x
```

```
## The deriviative is: 4.0
```

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Exact solution!

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It can take A LOT of programmer time, more than it is worth

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It can take A LOT of programmer time, more than it is worth

Humans are suseptible to error in coding or calculating the derivative mathematically

Automatic differentiation

Think about this: your code is **always** made up of simple arithmetic operations

- add, subtract, divide, multiply
- trig functions
- exponentials/logs
- etc

Automatic differentiation

Think about this: your code is **always** made up of simple arithmetic operations

- add, subtract, divide, multiply
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- etc

The closed form derivatives of these operations is not hard, it turns out your computer can do it and yield exact solutions

Automatic differentiation

How?

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How?

There are methods that basically apply a giant chain rule to your whole program, and break down the derivative into the (easy) component parts that another package knows how to handle

Automatic differentiation

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```
ff(x) = x^2
x = [2 3 4]
```

Automatic differentiation

```
using ForwardDiff
g(f,x) = ForwardDiff.derivative(f,x); # g = dff/dx
```

```
## g (generic function with 1 method)
```

```
println("ff'(x) at $(x) is: $(g.(ff,x))") # display gradient value
```

```
## ff'(x) at [2 3 4] is: [4 6 8]
```

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```
## ff'(x) at [2 3 4] is: [4 6 8]
```

Exact solutions without handcoding

Automatic differentiation

Once you get the hang of coding up function for autodiff it's not that hard

```
fff(x) = sin(x^2)
```

Automatic differentiation

Once you get the hang of coding up function for autodiff it's not that hard

```
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```

```
x = [0 1 2]
```

```
## 1×3 Matrix{Int64}:
##  0  1  2
```

```
println("fff'(x) at $(x) is: $(g.(fff,x))")
```

```
## fff'(x) at [0 1 2] is: [0.0 1.0806046117362795 -2.6145744834544478]
```

Calculus operations

Integration, trickier than differentiation

Calculus operations

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We integrate to do a lot of stuff in economics

Calculus operations

Integration, trickier than differentiation

We integrate to do a lot of stuff in economics

$\int_D f(x)dx, f : \mathcal{R}^n \to \mathcal{R}, D \subset \mathcal{R}^n$

How to think about integrals

Integrals are effectively infinite sums

How to think about integrals

Integrals are effectively infinite sums

1 dimensional example:

How to think about integrals

Integrals are effectively infinite sums

1 dimensional example:

$\lim_{dx_i \to 0} \sum_{i=0}^{(a-b)/dx_i} f(x_i) dx_i$

where dx_i is some subset of $[a, b]$ and x_i is some evaluation point (e.g. midpoint of dx_i)

Infinite limits strike again

Just like derivatives, we face an infinite limit as $(a - b)/dx_i \to \infty$

We avoid this issue in the same way as derivatives, we replace the infinite sum with something we can handle

Monte Carlo integration

Probably the most commonly used form in empirical econ

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Approximate an integral by relying on LLN and "randomly" sampling the integration domain

Monte Carlo integration

Probably the most commonly used form in empirical econ

Approximate an integral by relying on LLN and "randomly" sampling the integration domain

Can be effective for very high dimensional integrals

Very simple and intuitive

But, produces a random approximation

Monte Carlo integration

Suppose we want to integrate $\xi = \int_0^1 f(x)dx$

How do we do it?

Monte Carlo integration

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We can do so by drawing N uniformly distributed samples, $x_1, \ldots, x_N$ over interval $[0, 1]$

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Monte Carlo integration

ξ is equivalent to $E[f(x)]$ with respect to a uniform distribution, so estimating the integral is the same as estimating the expected value of $f(x)$

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where V is the volume over which we are integrating

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In general we have that $\hat{\xi} = V \frac{1}{N} \sum_{i=1}^{N} f(x_i)$

where V is the volume over which we are integrating

LLN gives us that the $plim_{N \to \infty} \hat{\xi} = \xi$

Monte Carlo integration

The variance of $\hat{\xi}$ is

$$\sigma^2_{\hat{\xi}} = \mathrm{var}\left(\frac{V}{N}\sum_{i=1}^{N} f(x_i)\right) = \frac{V^2}{N^2}\sum_{i=1}^{N}\mathrm{var}(f(X)) = \frac{V^2}{N}\sigma^2_{f(X)}$$

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The variance of $\hat{\xi}$ is

$$\sigma^2_{\hat{\xi}} = \mathrm{var}\left(\frac{V}{N}\sum_{i=1}^{N} f(x_i)\right) = \frac{V^2}{N^2}\sum_{i=1}^{N}\mathrm{var}(f(X)) = \frac{V^2}{N}\sigma^2_{f(X)}$$

So average error is $\frac{V}{\sqrt{N}}\sigma_{f(X)}$, this gives us its rate of convergence: $O(\sqrt{N})$

Note:

1. The rate of convergence is independent of the dimension of x
2. Quasi-Monte Carlo methods can get you $O(1/N)$

Monte Carlo integration

Suppose we want to integrate x^2 from 0 to 10, we know this is $10^3/3 = 333.333$

```
# Package for drawing random numbers
using Distributions

# Define a function to do the integration for an arbitrary function
function integrate_function(f, lower, upper, num_draws)

  # Draw from a uniform distribution
  xs = rand(Uniform(lower, upper), num_draws)

  # Expectation = mean(x)*volume
  expectation = mean(f(xs))*(upper - lower)

end
```

Monte Carlo integration

Suppose we want to integrate x^2 from 0 to 10, we know this is $10^3/3 = 333.333$

```
# Integrate
f(x) = x.^2;
integrate_function(f, 0, 10, 1000)
```

```
## 322.499755893713
```

Pretty close!

Quadrature rules

We can also approximate integrals using a technique called **quadrature**

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We will focus on two classes of quadrature for now:

1. Newton-Cotes (the kind you've seen before)
2. Gaussian (probably new)

Newton-Cotes quadrature rules

Suppose we want to integrate a one dimensional function $f(x)$ over $[a, b]$

How would you do it?

Newton-Cotes quadrature rules

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Key things to define up front:

- $x_i = a + (i-1)/h$ for $i = 1, 2, \ldots, n$ where $h = \frac{b-a}{n-1}$

x_is are the **quadrature nodes** of the approximation scheme and divide the interval into $n-1$ equally spaced subintervals of length h

Midpoint rule

The most basic Newton-Cotes method:

1. Split $[a, b]$ into intervals
2. Approximate the function in each subinterval by a constant equal to the function at the midpoint of the subinterval

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Approximates f by a step function

Trapezoid rule

Increase complexity by 1 degree:

1. Split $[a, b]$ into intervals
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We can aggregate this up to: $\int_a^b f(x)dx \approx \sum_{i=1}^{n} w_i f(x_i)$
where $w_1 = w_n = h/2$ and $w_i = h$ otherwise

How accurate is this rule?

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Seems sensible, a piecewise linear function can approximate any linear function exactly since it has more flexibility

Simpsons rule

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Let n be odd, then approximate the function across a **pair** of subintervals by a quadratic interpolation passing through $(x_{2i-1}, f(x_{2i-i}))$, $(x_{2i}, f(x_{2i}))$, and $(x_{2i+1}, f(x_{2i+1}))$

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We can aggregate this up to: $\int_a^b f(x)dx \approx \sum_{i=1}^n w_i f(x_i)$
where $w_1 = w_n = h/3$, otherwise and $w_i = 4h/3$ if i is even and $w_i = 2h/3$ if i is odd

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Why do we gain 2 orders of accuracy when increasing one order of approximation complexity?

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How accurate is this rule?

1. The approximating piecewise quadratic is exact at the end points and midpoint of the conjoined two subintervals
2. Clearly the difference between a cubic $f(x)$ and the quadratic approximation in $[x_{2i-1}, x_{2i+1}]$ is another cubic function
3. This cubic error is **odd** with respect to the midpoint: integrating over the first subinterval cancels integrating over the second subinterval, the integration error is zero

Gaussian quadrature rules

How did we pick the x_i quadrature nodes for Newton-Cotes rules?

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Evenly spaced, but no particular reason for doing so...

Gaussian quadrature selects these nodes more efficiently and relies on **weight functions** $w(x)$

Gaussian quadrature rules

Gaussian rules try to exactly integrate some finite dimensional collection of functions (i.e. polynomials up to some degree)

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For a given order of approximation n, the weights $w_1, \ldots, w_n$ and nodes $x_1, \ldots, x_n$ are chosen to satisfy $2n$ **moment-matching conditions**:

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For a given order of approximation n, the weights $w_1, \ldots, w_n$ and nodes $x_1, \ldots, x_n$ are chosen to satisfy $2n$ **moment-matching conditions**:

$\int_I x^k w(x) dx = \sum_{i=1}^n w_i x_i^k$, for $k = 0, \ldots, 2n-1$

where I is the interval over which we are integrating and $w(x)$ is a given weight function

Gaussian quadrature improves accuracy

The moment matching conditions pin down w_is and x_is so we can approximate an integral by a weighted sum of the function at the prescribed nodes

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$\int_i f(x)w(x)dx \approx \sum_{i=1}^{n} w_i f(x_i)$

Gaussian rules are $2n-1$ order exact: we can exactly compute the integral of any polynomial order $2n-1$

Gaussian quadrature takeaways

Gaussian quadrature effectively discretizes some distribution $p(x)$ into mass points (nodes) and probabilities (weights) for some other discrete distribution $\bar{p}(x)$

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Given an approximation with n mass points, X and $\bar{X}$ have identical moments up to order $2n-1$, and as $n \to \infty$ we have a continuum of mass points and recover the continuous pdf

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Given an approximation with n mass points, X and $\bar{X}$ have identical moments up to order $2n-1$, and as $n \to \infty$ we have a continuum of mass points and recover the continuous pdf

But what do we pick for the weighting function $w(x)$?

Gauss-Legendre

We can start out with a simple $w(x) = 1$, this gives us **Gauss-Legendre** quadrature

This can approximate the integral of any function arbitrarily well by increasing n

Gauss-Laguerre

Sometimes we want to compute exponentially discounted sums like:
$\int_I f(x)e^{-x}dx$

The weighting function e^{-x} is Gauss-Laguerre quadrature

Gauss-Hermite

Sometimes we want to take expectations of normally distributed variables:
$\int_I f(x)e^{-x^2}dx$

There exist packages or look-up tables to get the prescribed weights and nodes for each of these schemes

Linear Algebra

Lots of computational problems break down into linear systems

Many non-linear models are linearized

How do we **actually** solve these systems inside the machine?

L-U Factorization

If A in $Ax = b$ is upper or lower triangular, we can solve for x recursively via forward/backward substitution

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The equation in row 2 contains x_2 and the already solved for x_1
so we can easily solve for x_2 and then continue until we solve for all xs

Forward substitution

Forward substitution gives us solutions

$x_i = \frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j \right)$, for all i

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L-U factorization is an algorithm that decomposes A into the product of lower and upper triangular matrices

L-U Factorization has two steps

1. Factor A into lower L and upper U triangular matrices using Gaussian elimination
   - We can do this for any non-singular square matrix

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1. Factor A into lower L and upper U triangular matrices using Gaussian elimination
    - We can do this for any non-singular square matrix
2. Solve for x
    1. $(LU)x = b$
    2. Solve for y : $Ly = b$ using forward substitution
    3. Using the solved y, we know $Ux = y$ and can solve with backward substitution

Why bother with this scheme?

Why not just use another method like Cramer's rule?

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LU is less than $O(n^3)$

Cramer's rule is $O(n! \times n)$

For a 10x10 system this can really matter

Example: LU vs Cramer

Julia description of the division operator `\`: *If A is upper or lower triangular (or diagonal), no factorization of A is required and the system is solved with either forward or backward substitution. For non-triangular square matrices, an LU factorization is used.*

So we can do LU factorization approaches to solutions by just doing `x = A\b`, but we can write it ourselves as well

Example: LU vs Cramer

Cramer's Rule can be written as a simple loop:

```
function solve_cramer(A, b)

    dets = Vector(undef, length(b))

    for index in eachindex(b)
        B = copy(A)
        B[:, index] = b
        dets[index] = det(B)
    end

    return dets ./ det(A)

end
```

```
n = 100
A = rand(n, n)
b = rand(n)
```

Example: LU vs Cramer

Let's see the full results of the competition for a 10x10:

```
using BenchmarkTools
cramer_time = @elapsed solve_cramer(A, b);
cramer_allocation = @allocated solve_cramer(A, b);
lu_time = @elapsed A\b;
lu_allocation = @allocated A\b;

println("Cramer's rule solved in $cramer_time seconds and used $cramer_allocation kilobytes of m
LU solved in $(lu_time) seconds and used $(lu_allocation) kilobytes of memory.
LU is $(round(cramer_time/lu_time, digits = 0)) times faster and uses $(round(cramer_allocation/
```

```
## Cramer's rule solved in 0.01895975 seconds and used 16187200 kilobytes of memory.
## LU solved in 0.000146834 seconds and used 81840 kilobytes of memory.
## LU is 129.0 times faster and uses 198.0 times less memory.
```

Mechanics of factorizing

Gaussian elimination is where we use row operations

1. swapping rows
2. multiplying by non-zero scalars
3. add a scalar multiple of one row to another

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1. swapping rows
2. multiplying by non-zero scalars
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to turn a matrix $(I\,A)$ into $(L\,U)$

Numerical error blow up

Small errors can have big effects, for example:

$$\begin{bmatrix} -M^{-1} & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

where M is big

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$$\begin{bmatrix} -M^{-1} & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

where M is big

Lets use L-U Factorization to solve it:

$$\begin{bmatrix} -M^{-1} & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -M^{-1} & 1 \\ 1 & 1 \end{bmatrix}$$

Numerical error blow up

Subtract $-M$ times the first row from the second to get the L-U factorization

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We can get closed-form solutions by applying forward substitution:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} M/(M+1) \\ (M+2)/(M+1) \end{bmatrix}$$

Numerical issues

Both variables are approximately 1 for large M, but remember adding small numbers to big numbers causes problems numerically

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If $M = 10000000000000000000$, the computer will return x_2 is equal to precisely 1, this isn't terribly wrong

When we then perform the second step of backwards substitution, we solve for $x_1 = -M(1 - x_2) = 0$, this is **very** wrong

Large errors like this often occur because diagonal elements are very small

Julia example

```
function solve_lu(M)
    b = [1, 2]
    U = [-M^-1 1; 0 M+1]
    L = [1. 0; -M 1.]
    y = L\b
    # Round element-wise to 3 digits
    x = round.(U\y, digits = 5)
end;

true_solution(M) = round.([M/(M+1), (M+2)/(M+1)], digits = 5);
```

Julia example

```
## True solution for M=10 is approximately [0.90909, 1.09091], computed solution is [0.90909, 1.09091]

## True solution for M=1e10 is approximately [1.0, 1.0], computed solution is [1.0, 1.0]

## True solution for M=1e15 is approximately [1.0, 1.0], computed solution is [1.11022, 1.0]

## True solution for M=1e20 is approximately [1.0, 1.0], computed solution is [-0.0, 1.0]

## Julia's division operator is actually pretty smart though, true solution for M=1e20 is A\b = [1.0, 1
```

Ill-conditioning

A matrix A is said to be ill-conditioned if a small perturbation in b yields a large change in x

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One way to measure ill-conditioning in a matrix is the elasticity of the solution with respect to b,

$$\sup_{||\delta b||>0} \frac{||\delta x||/||x||}{||\delta b||/||b||}$$

which yields the percent change in x given
a percentage point change in the magnitude of b

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If this elasticity is large, then computer representations of the system of equations can lead to large errors due to rounding

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Approximate the elasticity by computing the **condition number**

$$\kappa = ||A|| \cdot ||A^{-1}||$$

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$$\kappa = ||A|| \cdot ||A^{-1}||$$

κ gives the least upper bound of the elasticity: it is always larger than one and a rule of thumb is that for every order of magnitude, a significant digit is lost in the computation of x

```
cond([1. 1.; 1. 1.00000001])
```

```
## 4.00000006235004454e8
```